Deductive systems with unified multiple-conclusion rules

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Thus, we need to answer the following questions:

(a) What do we prove?
(b) How do we prove it?

The brief answers are:
(a) We prove statements asserting or rejecting a given proposition;
(b) We use the multiple-conclusion rules which premises and conclusions are finite sets of statements.
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"The concepts of "truth", "falsehood", and "assertion" I owe to Frege. In adding "rejection" to "assertion" I have followed Brentano."

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According to Brentano and in contrast to Frege, assertion (or acceptance) and rejection (or refutation, or denial) should have the same status. Let us note that assertion of a negation is much stronger than the rejection. For instance, in the Classical Logic we reject formula \(p\) (in symbols \(\neg p\)), but the assertion \(\vdash \neg p\) does not hold.

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Łukasiewicz suggested to endow regular calculus (with rule of substitution) defining the Classical Logic (CPC), with the anti-axiom $\vdash p$ and the following two rules:

- **modus tolens:** $\vdash (A \rightarrow B), \vdash B / \vdash A$ (MT)
- **reversed substitution:** $\vdash \sigma(A) / \vdash A$ (Rs)

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Independently, Carnap suggested to include rejections into deductive systems: ”The rules of deduction usually consist of primitive sentences and rules of inference (defining 'directly definable in K'). Sometimes, K contains also rules of refutation (defining 'directly refutable in K').”\(^2\)

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Łukasiewicz suggested to endow regular calculus (with rule of substitution) defining the Classical Logic (CPC), with the anti-axiom \( \vdash p \) and the following two rules:

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Independently, Carnap suggested to include rejections into deductive systems: "The rules of deduction usually consist of primitive sentences and rules of inference (defining 'directly definable in K'). Sometimes, K contains also rules of refutation (defining 'directly refutable in K')."\(^2\) Moreover, Carnap also used the multiple-conclusion rules.

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Carnap’s motivation to introducing refutations and multiple-conclusion rules was requirement of categoricity: if we want to syntactically characterize two-valued classical semantics, this syntactical system should be valid only (up to matrix isomorphisms) on the two-element Boolean matrix. But any axiom and the rule which is valid in \((2, \{1\})\), is valid in all matrices \((2^n, \{1\})\), \(n \geq 0\) as well.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot \\
2^0 & 2 & 2^2 & 2^3 \\
\end{array}
\]
Carnap’s solution is to use refutations and multiple-conclusion (multiple-alternative) rules – the ordered pairs $\Gamma/\Delta$ of finite sets of formulas. In semantics, a rule $\Gamma/\Delta$ is valid in matrix $(A, D)$ if for any valuation $\nu$,

$$\nu(\Gamma) \subseteq D \text{ entails } \nu(\Delta) \cap D \neq \emptyset.$$ 

A rejected (refuted) proposition $\neg A$ is valid in a given matrix, if for some valuation, the value of $A$ is not designated. For instance, $\neg p$, where $p$ is a variable, is valid in any matrix having at least one non-designated element, and $\neg p$ is invalid in all matrices in which every element is designated.
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Before we proceed, the warning:

We do not consider multiple-conclusion logics in the sense of Shoesmith and Smiley or Carnap’s junctives.

We use multiple-conclusion rules merely as means of derivation of a statement from a set of statements.
Outline

Introduction

Unified Logic

Multiple-Conclusion Rules
Unified Logic

We assume that Frm is a set of propositional formulas built in a regular way from a countable set $\text{Var}$ of propositional variables and a finite set of connectives $\Omega$.

**Definition**

A *unified logic* is an ordered pair $(L^+, L^-)$, where $L^+$ is a set of formulas closed under the rule of substitution: $\text{Sb} := A/\sigma(A)$, where $A \in \text{Frm}$ and $\sigma$ is a substitution, while $L^-$ is a set of formulas closed under the rule of reverse substitution: $\text{Rs} := \sigma(A)/A$. 
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For example, let \(\text{Cl}^+\) be a set of all classical tautologies and \(\text{Cl}^- := \text{Frm} \setminus \text{Cl}^+\). Then the pair \(\text{UCL} := (\text{Cl}^+, \text{Cl}^-)\) is a unified classical logic.
L^+ is a set of \textit{asserted} (accepted) propositions – \textit{theorems} ; \ L^- is a set of \textit{rejected} (refuted, denied) propositions – \textit{anti-theorems} ;
Unified Logic

$L^+$ is a set of \textit{asserted} (accepted) propositions – \textit{theorems} ; $L^-$ is a set of \textit{rejected} (refuted, denied) propositions – \textit{anti-theorems} ;

We make no assumptions regarding relations between $L^+$ and $L^-$. All possibilities are admissible:

$L^+$ - asserted propositions
$L^-$ - rejected propositions
$L^\land$ - asserted and rejected propositions
$L^\circ$ - neither asserted nor rejected propositions
Introduction: Types of unified logics

If $L^+ \cap L^- = \emptyset$, the logic is **coherent**.
Introduction: Types of unified logics

If $L^+ \cap L^- = \emptyset$, the logic is *coherent*.

If $L^+ \cup L^- = \text{Frm}$, the logic is *full*.
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A full and coherent logic is called *standard*. 
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A full and coherent logic is called **standard**.

**Example.** Let us take the three-element Heyting algebra \( A := (\{0, a, 1\}; \to, \land, \lor, \neg) \), and consider a (logical) matrix \( M := (A; D^+ = \{1\}, D^- = \{0\}) \). For any proposition \( A \), we let

\[
A \in L^+ \iff \text{for each valuation } \nu, \nu(A) \in D^+;
\]

\[
A \in L^- \iff \text{there is a valuation } \nu, \text{ such that } \nu(A) \in D^-.
\]

Then, \( A \in L^+ \) if and only if \( A \) is valid in the Smetanich logic. \( A \in L^- \) if and only is \( A \) is invalid in the Classical logic. Propositions \( p \lor \neg p \) and \( \neg \neg p \to p \) are neither asserted, not rejected.
Unified Logic

It is custom to define logic by a consequence relation. If assertions and rejections have the same status, we need to consider the consequence relations on sets of meta-statements of the type "$A$ is asserted" ($A \in L^+$) and "$A$ is rejected" ($A \in L^-$).

\footnote{The similar notations are used in T. Smiley "Rejection 1996 and I. Rumfitt "Yes and No", 2000.}
Unified Logic

It is custom to define logic by a consequence relation. If assertions and rejections have the same status, we need to consider the consequence relations on sets of meta-statements of the type ”A is asserted” \((A \in L^+)\) and ”A is rejected” \((A \in L^-)\).

It is inconvenient for our purposes to use \(\vdash\) and \(\dashv\) for ”is asserted” and ”is rejected”, because the notation like

\[
\vdash A_1, \ldots, \vdash A_n \vdash \vdash B
\]

looks confusing. Instead, we use \(\oplus A\) for ”A is asserted”, and \(\ominus A\) for ”A is rejected”. The notation

\[
\oplus A_1, \ldots, \oplus A_n \vdash \oplus B
\]

is less confusing.\(^3\)

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Statements

*Meta-statements* (or statements, for short) are expressions of form $\oplus A$ – *positive* or assertions, and $\ominus A$ – *negative* or rejections, where $A \in \text{Frm}$.
Statements

*Meta-statements* (or statements, for short) are expressions of form \( \Theta A \) – *positive* or assertions, and \( \Theta A \) – *negative* or rejections, where \( A \in \text{Frm} \). The set of all statements is denoted by \( S \), and by \( S^+ \) and \( S^- \) we denote respectively the set of all positive and the set of all negative statements.
Statements

Meta-statements (or statements, for short) are expressions of form $\oplus A$ – positive or assertions, and $\ominus A$ – negative or rejections, where $A \in \text{Frm}$. The set of all statements is denoted by $S$, and by $S^+$ and $S^-$ we denote respectively the set of all positive and the set of all negative statements.

Unified consequence relation is a binary relation $\vdash$ defined on sets of statements and statements and satisfying the regular properties of consequence relation: for any sets $\Gamma, \Delta \subseteq S$ and any $\alpha, \beta \in S$

\[
\begin{align*}
\alpha \vdash \alpha & \quad (R) \\
\text{if } \Gamma \vdash \alpha \text{ and } \Gamma \subseteq \Delta, \text{ then } & \Delta \vdash \alpha & \quad (M) \\
\text{if } \Gamma \vdash \alpha \text{ and } \alpha, \Delta \vdash \beta, \text{ then, } & \Gamma, \Delta \vdash \beta & \quad (T)
\end{align*}
\]
Meta-statements (or statements, for short) are expressions of form $\oplus A$ – positive or assertions, and $\ominus A$ – negative or rejections, where $A \in \text{Frm}$. The set of all statements is denoted by $S$, and by $S^+$ and $S^-$ we denote respectively the set of all positive and the set of all negative statements.

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$$\alpha \vdash \alpha \quad (R)$$

if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha \quad (M)$

if $\Gamma \vdash \alpha$ and $\alpha, \Delta \vdash \beta$, then, $\Gamma, \Delta \vdash \beta. \quad (T)$

Relation $\vdash$ is finitary if $\Gamma \vdash \alpha$ entails $\Gamma' \vdash \alpha$ for some finite $\Gamma' \subseteq \Gamma$. 

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Logic: theorems

Each unified consequence relation \( \vdash \) defines a set of *asserting theorems*:

\[
\text{Th}^+(\vdash) := \{ \alpha \in S^+ \mid \vdash \alpha \}
\]

and a set of *refuting theorems*:

\[
\text{Th}^-(\vdash) := \{ \alpha \in S^- \mid \vdash \alpha \}.
\]

The set \( \text{Th}(\vdash) := \text{Th}^+(\vdash) \cup \text{Th}^-(\vdash) \) is a set of *theorems*. 
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But if we consider the projections onto the set of propositions:

$$\text{L}^+ := \{ A \in \text{Frm} \mid \bigoplus A \in \text{Th}^+(\vdash) \},$$

$$\text{L}^- := \{ A \in \text{Frm} \mid \bigotimes A \in \text{Th}^-(\vdash) \}$$

the situation is different.
Introduction: types of refutation

In general, there are two ways of how to handle refutation syntactically: direct and indirect. To determine whether formula $A$ is refutable one can do one of two things:

(a) to derive in a meta-logic a statement about refutability of $A$ ($\mathcal{L}$-proof - Łukasiewicz-style proof)

\[\text{\textsuperscript{4}}\text{W. Staszek ”On Proofs and Rejections”, 1971}\]
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(b) to derive from $A$ a formula $B$ that we already know is refutable, and apply Modus Tollens (i-proof - indirect proof, Carnap’s way)

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An existence of an $\mathcal{L}$-proof entails the existence of $i$-proof. The converse is true under some assumptions (some weak form of the deduction theorem$^4$) and we will revisit this issue later.

$^4$W. Staszek ”On Proofs and Rejections”, 1971
As an example, let us consider a Classical Propositional Calculus (CPC) with regular set of axioms and rules

\[ \oplus A, \vdash \oplus (A \rightarrow B)/ \oplus B \]  
\[ \oplus A/ \oplus \sigma(A), \text{ where } \sigma \text{ is a substitution} \]  

(MP)

(Sb)
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\[ \oplus A/ \oplus \sigma(A), \text{ where } \sigma \text{ is a substitution} \quad \text{(Sb)} \]

And let us extend this calculus to calculus $\text{CPC}^\circ$ by adding an anti-axiom

\[ \vdash \ominus p, \]

where $p$ is a propositional variable, and two rules

\[ \ominus(A \to B), \ominus B/ \ominus A \quad \text{(MT)} \]
\[ \ominus \sigma(A)/ \ominus A, \text{ where } \sigma \text{ is a substitution} \quad \text{(Rs)} \]
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Let $A$ be a formula invalid in CPC. Then, there is a substitution $\sigma$ such that $\neg \sigma(A)$ is valid in CPC. Hence, in CPC (and CPC°) we have

$$\vdash \sigma(A) \rightarrow p \text{ or } \vdash \oplus(\sigma(A) \rightarrow p) \text{ in CPC°}.$$
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Therefore, by (MT), we have

\[ \vdash \Theta \sigma(A) \]

and, by (Rs), we have

\[ \vdash \Theta A. \]
Łukasiewicz has observed that $\text{CPC}^\circ$ is a complete axiomatization for classical logic. It is clear that every classically valid formula is derivable in $\text{CPC}^\circ$.

Let $A$ be a formula invalid in $\text{CPC}$. Then, there is a substitution $\sigma$ such that $\neg \sigma(A)$ is valid in $\text{CPC}$. Hence, in $\text{CPC}$ (and $\text{CPC}^\circ$) we have

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Therefore, by (MT), we have

$$\vdash \ominus \sigma(A)$$

and, by (Rs), we have

$$\vdash \ominus A.$$  

Soundness easily follows from the observation that all axioms, the anti-axiom and the rules are valid in the 2-element Boolean algebra.
**Introduction: Direct refutation, an example**

Let us take any intermediate logic $\mathcal{I}$ – a logic extending IPC and contained in CPC, and add the anti-axiom $\vdash \Theta \neg p$ and the rules MT and Rs. In such a way we obtain a unified logic $I^\circ$. 
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For any formula $A$, if $I \vdash A$, then $I^\circ \vdash \Theta A$. 
Let us take any intermediate logic \( \mathcal{I} \) – a logic extending IPC and contained in CPC, and add the anti-axiom \( \vdash \Theta p \) and the rules MT and Rs. In such a way we obtain a unified logic \( \mathcal{I}^\circ \).

For any formula \( A \), if \( \mathcal{I} \vdash A \), then \( \mathcal{I}^\circ \vdash \Theta A \).

If \( \mathcal{I} \nvdash \neg\neg A \), then, \( A \) is invalid in CPC, and we can repeat the argument used for CPC and conclude that \( \mathcal{I}^\circ \vdash \Theta A \).
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For any formula \( A \), if \( \mathcal{I} \vdash A \), then \( \mathcal{I}^\circ \vdash \oplus A \).

If \( \mathcal{I} \nvdash \neg \neg A \), then, \( A \) is invalid in CPC, and we can repeat the argument used for CPC and conclude that \( \mathcal{I}^\circ \vdash \ominus A \).

We can use the semantic means and conclude that

\[
L^+ (\mathcal{I}^\circ) = \{ A \in \text{Frm} \mid \mathcal{I} \vdash A \},
\]

\[
L^- (\mathcal{I}^\circ) = \{ A \in \text{Frm} \mid \mathcal{I} \nvdash \neg \neg A \} = \{ A \in \text{Frm} \mid CPC \nvdash A \}.
\]
Let us take any intermediate logic $\mathcal{I}$ – a logic extending IPC and contained in CPC, and add the anti-axiom $\vdash \Theta p$ and the rules MT and Rs. In such a way we obtain a unified logic $\mathcal{I}^\circ$.

For any formula $A$, if $\mathcal{I} \vdash A$, then $\mathcal{I}^\circ \vdash \Theta A$.

If $\mathcal{I} \not\vdash \neg\neg A$, then, $A$ is invalid in CPC, and we can repeat the argument used for CPC and conclude that $\mathcal{I}^\circ \vdash \Theta A$.

We can use the semantic means and conclude that

\[
L^+(\mathcal{I}^\circ) = \{ A \in \text{Frm} \mid \mathcal{I} \vdash A \}, \quad L^-(\mathcal{I}^\circ) = \{ A \in \text{Frm} \mid \mathcal{I} \not\vdash \neg\neg A \} = \{ A \in \text{Frm} \mid CPC \not\vdash A \}.
\]

If $\mathcal{I} \not\vdash A$ and $\mathcal{I} \not\vdash \neg\neg A$, then $\mathcal{I}^\circ \not\vdash \Theta A$ and $\mathcal{I}^\circ \not\vdash \Theta A$. Thus,

\[
L^+(\mathcal{I}^\circ) \cup L^-(\mathcal{I}^\circ) \neq \text{Frm},
\]

that is, $\mathcal{I}$ is not full.
Outline

Introduction

Unified Logic

Multiple-Conclusion Rules
Multiple-Alternative Rules

If \( \Gamma, \Delta \) are finite sets of meta-statements, an ordered pair \( \Gamma/\Delta \) is called a *structural multiple-conclusion* or *multiple-alternative* rule (m-rule for short). The premises \( \Gamma \) are viewed conjunctively, while the conclusions \( \Delta \) are viewed disjunctively.
Multiple-Alternative Rules

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In general, we divide rules into three categories: if $r := \Gamma/\Delta$ is a rule, then

- $r$ is *conclusive* if $\Delta$ consists of a single formula
- $r$ is *inconclusive* if $\Delta$ consists of more than one formula
- $r$ is *terminating* if $\Delta = \emptyset$
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In general, we divide rules into three categories: if \( r := \Gamma/\Delta \) is a rule, then

- \( r \) is **conclusive** if \( \Delta \) consists of a single formula
- \( r \) is **inconclusive** if \( \Delta \) consists of more then one formula
- \( r \) is **terminating** if \( \Delta = \emptyset \)

For instance, \( \oplus p, \oplus(p \rightarrow q)/\oplus q \) is a conclusive rule; \( \oplus(p \lor q)/\oplus p, \oplus q \) is an inconclusive rule; \( \oplus p, \ominus p/\emptyset \) is a terminating rule.

In addition to m-rules, we consider two rules: the rule of substitution Sb, and the rule of reverse substitution Rs.
Multiple-Alternative Rules

Multiple-alternative rules allow to explicitly use the proofs by cases.
Multiple-Alternative Rules

Multiple-alternative rules allow to explicitly use the proofs by cases. In the setting of natural deduction, proof by cases looks like this:

\[
\begin{array}{c}
[A] \\
\vdots \\
A \lor B \\
\end{array} \quad \begin{array}{c}
[B] \\
\vdots \\
C \\
\end{array} \\
\hline
\begin{array}{c}
C \\
C \\
\end{array}
\]

By applying rule \( \Gamma /\Gamma \) we get the alternatives \( \Delta \) to be considered separately.
Multiple-Alternative Rules

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\[
\begin{array}{c}
[A] \\
[\vdots] \\
A \lor B \\
\vdash C \\
[\vdots] \\
B \\
\vdash C \\
\end{array}
\]

\[\frac{A \lor B}{C} \quad \frac{C}{C}\]

In the multiple-alternative setting, proof by cases looks like this:

\[
\begin{array}{c}
(A \lor B) \\
\vdash (p \lor q)/p, q \\
\vdash A \\
\vdash B \\
\vdash C \\
\vdash C \\
\end{array}
\]
Multiple-Alternative Rules

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\[
\begin{array}{c}
[A] \quad [B] \\
\vdots \quad \vdots \\
A \lor B \quad C \quad C \\
\hline
C
\end{array}
\]

In the multiple-alternative setting, proof by cases looks like this:

\[
\begin{array}{c}
A \lor B \\
\hline
(p \lor q)/p, q \\
A \quad B \\
\vdots \quad \vdots \\
C \quad C
\end{array}
\]

By applying rule \( \Gamma/\Delta \) we get the alternatives \( \Delta \) to be considered separately.
We use ▼ to denote an empty set of premises, and ▲ to denote an empty set of alternatives. ▼ and ▲ are merely notations and they are not the symbols of the language or meta-language.

\footnote{Note that we define an inference from \((\Gamma, R)\) without clarifying what we are deriving.}
Multiple-Alternative Inference

We use $\nabla$ to denote an empty set of premises, and $\triangledown$ to denote an empty set of alternatives. $\nabla$ and $\triangledown$ are merely notations and they are not the symbols of the language or meta-language.

Inferences are finite trees the nodes of which are labeled by statements, $\nabla$ or $\triangledown$. A leaf labeled by $\triangledown$ is *terminating* (we have reduced a case to contradiction), otherwise, it is *extendable*.

---

Note that we define an inference from $(\Gamma, R)$ without clarifying what we are deriving.
Multiple-Alternative Inference

We use \( \nabla \) to denote an empty set of premises, and \( \triangle \) to denote an empty set of alternatives. \( \nabla \) and \( \triangle \) are merely notations and they are not the symbols of the language or meta-language.

Inferences are finite trees the nodes of which are labeled by statements, \( \nabla \) or \( \triangle \). A leaf labeled by \( \triangle \) is terminating (we have reduced a case to contradiction), otherwise, it is extendable.

Let \( R \) be a set of rules (that may include Sb and/or Rs) and \( \Gamma \) be a set of statements (which may be empty). An inference from \( \Gamma \) by \( R \) (or \( (\Gamma, R) \)-inference for short) is a finite tree nodes of which are labeled by statements, and it is defined by induction:\(^5\):

---

\(^5\)Note that we define an inference from \( (\Gamma, R) \) without clarifying what we are deriving.
Multiple-Alternative Inference

Like in a Hilbert-style inference, we use the assumptions and apply the inference rules.

A tree consisting of a single node (a root) labeled by $\bowtie$ is a $(\Gamma, R)$-inference (*it is needed for a sake of convenience*).
Multiple-Alternative Inference

Like in a Hilbert-style inference, we use the assumptions and apply the inference rules.

A tree consisting of a single node (a root) labeled by $\triangledown$ is a $(\Gamma, R)$-inference (it is needed for a sake of convenience).

**Using the assumptions:** if $I$ is a $(\Gamma, R)$-inference, then any non-terminal leaf can be extended by adjoining a leaf labeled by a statement from $\Gamma$, and the obtained tree is a $(\Gamma, R)$-inference.
Multiple-Alternative Inference

Like in a Hilbert-style inference, we use the assumptions and apply the inference rules.

A tree consisting of a single node (a root) labeled by \( \nabla \) is a \((\Gamma, R)\)-inference (*it is needed for a sake of convenience*).

*Using the assumptions*: if \( \mathcal{I} \) is a \((\Gamma, R)\)-inference, then any non-terminal leaf can be extended by adjoining a leaf labeled by a statement from \( \Gamma \), and the obtained tree is a \((\Gamma, R)\)-inference.

*Applying the rules*: if \( \mathcal{I} \) is a \((\Gamma, R)\)-inference, then any non-terminal leaf \( \lambda \) can be extended by adjoining the leaves labeled by \( \triangleright \), or by statements from a finite set \( \Delta \), provided there is an instance \( \Xi/\triangleright \) or \( \Xi/\Delta \) of a rule from \( R \), and all statements from \( \Xi \) are between \( \lambda \) and the root. The tree obtained in such a way is a \((\Gamma, R)\)-inference.
Multiple-Alternative Inference

Suppose that $\frac{\xi_1, \ldots, \xi_m}{\delta_1, \ldots, \delta_n}$ is an instance of a rule from $R$. 

\[ \begin{array}{ccc}
\alpha_1 & \alpha_2 & \ldots \\
\downarrow & & \\
\xi_1 & & \\
\downarrow & & \\
\xi_m & \rightarrow & \\
\downarrow & & \\
\alpha_k & & \\
\downarrow & & \\
\delta_1 & \ldots & \delta_n \\
\end{array} \]

Alex Citkin

Deductive systems with unified multiple-conclusion rules
Multiple-Alternative Inference

Let $\Gamma, \Delta$ be sets of statements, $\alpha$ be a statement and $R$ be a set of rules.

**Definition**

$\alpha$ is derivable from $\Delta$ by $(\Gamma, R)$, if there is a $(\Delta \cup \Gamma, R)$-inference each leaf of which is labeled by $\alpha$ or by $\triangledown$.
**Multiple-Alternative Inference**

Let $\Gamma, \Delta$ be sets of statements, $\alpha$ be a statement and $R$ be a set of rules.

**Definition**

$\alpha$ is derivable from $\Delta$ by $(\Gamma, R)$, if there is a $(\Delta \cup \Gamma, R)$-inference each leaf of which is labeled by $\alpha$ or by $\triangledown$.

Roughly speaking, $\alpha$ is derivable from $\Delta$ if after we have considered every case arisen in the proof, we either have derived $\alpha$, or we have arrived at a contradiction, meaning, that the case is not possible.
Multiple-Alternative Inference

Let $\Gamma, \Delta$ be sets of statements, $\alpha$ be a statement and $R$ be a set of rules.

Definition

$\alpha$ is derivable from $\Delta$ by $(\Gamma, R)$, if there is a $(\Delta \cup \Gamma, R)$-inference each leaf of which is labeled by $\alpha$ or by $\triangle$.

Roughly speaking, $\alpha$ is derivable from $\Delta$ if after we have considered every case arisen in the proof, we either have derived $\alpha$, or we have arrived at a contradiction, meaning, that the case is not possible.

Proposition

Any pair consisting of a set of statements $\Gamma$ and a set of rules $R$, defines a consequence relation:

$$\Delta \vdash \alpha \iff \alpha \text{ is derivable from } \Delta \text{ by } (\Gamma, R).$$
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

Жаба прошлепала большим ртом:
- Пациент скорее жив, чем мертв...
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

Жаба прошлепала большим ртом:
- Пациент скорее жив, чем мертв...

- Одно из двух, - прошелестел Народный лекарь Богомол, - или пациент жив, или он умер. Если он жив - он останется жив или он не останется жив. Если он мертв - его можно оживить или нельзя оживить.
Multiple-Alternative vs. Classical Inference: an Example

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Девочка всплеснула хорошенькими руками:
- Ну, как же мне его лечить, граждане?
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

Жаба прошлепала большим ртом:
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Девочка всплеснула хорошенькими руками:
- Ну, как же мне его лечить, граждане?
- Касторкой, - квакнула Жаба.
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

Жаба прошлепала большим ртом:
- Пациент скорее жив, чем мертв...

- Одно из двух, - прошелестел Народный лекарь Богомол, - или пациент жив, или он умер. Если он жив - он останется жив или он не останется жив. Если он мертв - его можно оживить или нельзя оживить.

Девочка всплеснула хорошенькими руками:
- Ну, как же мне его лечить, граждane?
- Касторкой, - квакнула Жаба.
- Касторкой! - презрительно захохотала Сова.
Multiple-Alternative vs. Classical Inference: an Example

Сова приложила ухо к груди Буратино.
- Пациент скорее мертв, чем жив, - прошептала она.

Жаба прошепала большим ртом:
- Пациент скорее жив, чем мертв...

- Одно из двух, - прошелестел Народный лекарь Богомол, - или пациент жив, или он умер. Если он жив - он останется жив или он не останется жив. Если он мертв - его можно оживить или нельзя оживить.

Девочка всплеснула хорошенькими руками:
- Ну, как же мне его лечить, гражданин?
- Касторкой, - квакнула Жаба.
- Касторкой! - презрительно захохотала Сова.
- Или касторкой, или не касторкой, - проскрежетал Богомол.
**Multiple-Alternative Inference**

**Definition**

Unified deductive system is a pair \((\Gamma, R)\), where \(\Gamma\) is a set (maybe empty) of statements, and \(R\) is a set (maybe empty) of rules.
**Definition**

Unified deductive system is a pair \((\Gamma, R)\), where \(\Gamma\) is a set (maybe empty) of statements, and \(R\) is a set (maybe empty) of rules.

Let \(L = (L^+, L^-)\) be a unified logic. A deductive system \(\mathcal{D}\) is \(L\)-complete for \(L\), or \(L\) is defined by \(\mathcal{D}\), if

\[
\vdash_{\mathcal{D}} \alpha \iff \begin{cases} 
\alpha = \bigoplus A \text{ and } A \in L^+ \\
\alpha = \bigominus A \text{ and } A \in L^-.
\end{cases}
\]

Any unified deductive system defines a unified logic.
Multiple-Alternative Inference

Definition

Unified deductive system is a pair \((\Gamma, R)\), where \(\Gamma\) is a set (maybe empty) of statements, and \(R\) is a set (maybe empty) of rules.

Let \(L = (L^+, L^-)\) be a unified logic. A deductive system \(D\) is \(L\)-complete for \(L\), or \(L\) is defined by \(D\), if

\[
\vdash_D \alpha \iff \begin{cases} 
\alpha = \oplus A \text{ and } A \in L^+ \\
\alpha = \ominus A \text{ and } A \in L^-.
\end{cases}
\]

Any unified deductive system defines a unified logic.

If \(D\) contains only positive rules, it is \(C\)-complete for \(L\), if

\[
\vdash_D \alpha \iff \begin{cases} 
\alpha = \oplus A \text{ and } A \in L^+ \\
\alpha = \ominus A \text{ and } \ominus A \vdash_D \ominus B, \text{ where } \ominus B \text{ is an anti-axiom.}
\end{cases}
\]
Admissible multiple-alternative rules

An m-rule $\Gamma/\Delta$ is *admissible* for a given unified logic $L$, if for each substitution that makes valid all statements from $\Gamma$, at least one statement from $\Delta$ is valid. $\nabla$ is considered being always valid, and $\triangledown$ is considered being always invalid.
Admissible multiple-alternative rules

An m-rule $\Gamma/\Delta$ is *admissible* for a given unified logic $L$, if for each substitution that makes valid all statements from $\Gamma$, at least one statement from $\Delta$ is valid. $\blacklozenge$ is considered being always valid, and $\blacklozenge$ is considered being always invalid.

For instance, a rule $\Gamma/\blacklozenge$ is admissible for $\vdash$ if and only if neither substitution makes valid all statements from $\Gamma$. 
Admissible multiple-alternative rules

An m-rule $\Gamma/\Delta$ is *admissible* for a given unified logic $L$, if for each substitution that makes valid all statements from $\Gamma$, at least one statement from $\Delta$ is valid. $\top$ is considered being always valid, and $\bot$ is considered being always invalid.

For instance, a rule $\Gamma/\bot$ is admissible for $\vdash$ if and only if neither substitution makes valid all statements from $\Gamma$.

**Proposition**

*In any intermediate logic, for any formula $A$,*

$$\text{rule } \oplus A/\bot \text{ is admissible if and only if rule } \bot/\oplus \neg A \text{ is admissible.}$$

The proof of $\Leftarrow$ is trivial, while $\Rightarrow$ follows immediately from the Glivenko Theorem.
Admissible multiple-alternative rules

In terms of admissible rules, we have the following:

*(coherency)* a logic is coherent if and only if the rule

\[ \text{Co} := \frac{\bigoplus p, \bigoplus p}{\blacktriangleleft} \text{is admissible;} \]

*(fullness)* a logic is full if and only if the rule

\[ \text{Fu} := \frac{\bigoplus p, \bigoplus p}{\blacktriangledown} \text{is admissible.} \]
Admissible multiple-alternative rules

In terms of admissible rules, we have the following:

(coherency) a logic is coherent if and only if the rule

\[ \text{Co} := \frac{\ominus p, \ominus p}{\downarrow} \text{ is admissible; } \]

(fullness) a logic is full if and only if the rule

\[ \text{Fu} := \frac{\ominus p, \ominus p}{\ominus p, \ominus p} \text{ is admissible. } \]

In what follows, the above m-rules play the central role.
Admissible multiple-alternative rules

In terms of admissible rules, we have the following:

(coherency) a logic is coherent if and only if the rule

\[ \text{Co} := \frac{\ominus p, \ominus p}{\updownarrow} \] is admissible;

(fullness) a logic is full if and only if the rule

\[ \text{Fu} := \frac{\ominus p, \ominus p}{\oplus p, \ominus p} \] is admissible.

In what follows, the above m-rules play the central role.

For convenience, we use the notation:

\[ \overline{\alpha} = \begin{cases} \ominus A, & \text{when } \alpha = \ominus A \\ \oplus A, & \text{when } \alpha = \ominus A. \end{cases} \]
Admissible multiple-alternative rules

Let $L$ be a standard logic. Then, the following holds: for any finite sets $\Gamma, \Delta$ and any statement $\alpha$,

- if the rule \( \frac{\alpha, \Gamma}{\Delta} \) is admissible, then the rule \( \frac{\Gamma}{\alpha, \Delta} \) is admissible;
- if the rule \( \frac{\Gamma}{\alpha, \Delta} \) is admissible, then the rule \( \frac{\overline{\alpha}, \Gamma}{\Delta} \) is admissible.

In other words, one can move a statement from premises to alternatives, or vice-versa, with changing the “sign” of the statement.

For logics without rejection the above makes no sense.
Admissible multiple-alternative rules

Let L be a standard logic. Then, the following holds: for any finite sets \( \Gamma, \Delta \) and any statement \( \alpha \),

1. If the rule \( \frac{\alpha, \Gamma}{\Delta} \) is admissible, then the rule \( \frac{\Gamma}{\overline{\alpha}, \Delta} \) is admissible;
2. If the rule \( \frac{\Gamma}{\alpha, \Delta} \) is admissible, then the rule \( \frac{\overline{\alpha}, \Gamma}{\Delta} \) is admissible.

In other words, one can move a statement from premises to alternatives, or vice-versa, with changing the "sign" of the statement.
Let $L$ be a standard logic. Then, the following holds: for any finite sets $\Gamma, \Delta$ and any statement $\alpha$, 

- if the rule $\frac{\alpha, \Gamma}{\Delta}$ is admissible, then the rule $\frac{\Gamma}{\bar{\alpha}, \Delta}$ is admissible;
- if the rule $\frac{\Gamma}{\alpha, \Delta}$ is admissible, then the rule $\frac{\bar{\alpha}, \Gamma}{\Delta}$ is admissible.

In other words, one can move a statement from premises to alternatives, or vice-versa, with changing the "sign" of the statement. For logics without rejection the above makes no sense.
Admissible multiple-alternative rules

Let L be a standard logic signature of which contains $\rightarrow$. If Modus Ponens is admissible for L, then, all the following eight variations of Modus Ponens are admissible:

$$
\begin{align*}
\nabla &\: \theta p, \theta(p \rightarrow q), \oplus q; & \oplus p \: \theta(p \rightarrow q), \oplus q; & \oplus(p \rightarrow q) \: \theta p, \theta q; & \theta p, \theta(p \rightarrow q), \theta q; \\
&\downarrow & \uparrow & \uparrow & \uparrow \\
\n\end{align*}
$$
Admissible multiple-alternative rules

Let $L$ be a standard logic signature of which contains $\rightarrow$. If Modus Ponens is admissible for $L$, then, all the following eight variations of Modus Ponens are admissible:

$$
\begin{align*}
\downarrow & \quad \uplus p \quad \uplus (p \rightarrow q) \quad \uplus q \\
\Theta p, \Theta (p \rightarrow q), \uplus q' & \quad \Theta (p \rightarrow q), \uplus q' \\
\Theta p, \Theta q & \quad \Theta p, \Theta (p \rightarrow q),
\end{align*}
$$

By the same argument, for the rule of substitution we have two variations that are either simultaneously admissible, or simultaneously not admissible:

$$
\begin{align*}
\uplus A & \quad \Theta \sigma (A) \\
\uplus \sigma (A) & \quad \Theta A
\end{align*}
$$
Derivations of rules

Let $R$ be a set of rules and $r := \Gamma/\Delta$ be a rule. We say that $r$ is derivable from $R$ (in symbols $R \vdash r$), if there is a $(\Gamma, R)$-inference all leaves of which do not contain statements not from $\Delta$. 

Alex Citkin

Deductive systems with unified multiple-conclusion rules
Derivations of rules

Let $R$ be a set of rules and $r := \Gamma/\Delta$ be a rule. We say that $r$ is derivable from $R$ (in symbols $R \vdash r$), if there is a $(\Gamma, R)$-inference all leaves of which do not contain statements not from $\Delta$.

If $R$ is a set of rules and $r, r'$ are rules, we say that $r$ is derivable from $r'$ relative to $R$ (in symbols $r' \vdash_R r$), if $R, r' \vdash r$. 
Derivations of rules

Let $R$ be a set of rules and $r := \Gamma/\Delta$ be a rule. We say that $r$ is **derivable** from $R$ (in symbols $R \vdash r$), if there is a $(\Gamma, R)$-inference all leaves of which do not contain statements not from $\Delta$.

If $R$ is a set of rules and $r, r'$ are rules, we say that $r$ is **derivable from** $r'$ **relative** to $R$ (in symbols $r' \vdash_R r$), if

$$R, r' \vdash r.$$ 

$r' \vdash_R r$ means that in any inference, every application of rule $r$ can be replaced with the suitable applications of rules $R$ and $r'$. In other words, rule $r$ can be eliminated from any inference and replaced by rules $R, r'$. 
Derivations of rules

Let $R$ be a set of rules and $r := \Gamma/\Delta$ be a rule. We say that $r$ is derivable from $R$ (in symbols $R \vdash r$), if there is a $(\Gamma, R)$-inference all leaves of which do not contain statements not from $\Delta$.

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$r' \vdash_R r$ means that in any inference, every application of rule $r$ can be replaced with the suitable applications of rules $R$ and $r'$. In other words, rule $r$ can be eliminated from any inference and replaced by rules $R, r'$.

The rules Co and Fu allows to derive the different variations of the given rules from each other. Let

$$S := \{Co, Fu\}.$$
Reduction of Rs to Sb

Proposition. \( Sb \vdash S Rs. \)
Reduction of Rs to Sb

**Proposition.** \( \text{Sb} \vdash_{S} \text{Rs} \).

Thus, in each deductive system that has postulated rules Co, Fu and Sb, the rule Rs can be eliminated.
Proposition. \( Sb \vdash_s \text{Rs} \).

Thus, in each deductive system that has postulated rules \( \text{Co} \), \( \text{Fu} \) and \( Sb \), the rule \( \text{Rs} \) can be eliminated.
**Proposition.** \( \text{MP} \vdash_{S} \text{MT} \).
Theorem

Let $\mathcal{D}$ be a deductive system containing only positive rules and the rule of substitution. Then, if $\mathcal{D}$ is $C$-complete for a unified logic $L$, the system $\mathcal{D}'$ obtained from $\mathcal{D}$ by postulating $Co$ and $Fu$, is $L$-complete.
**Theorem**

Let $\mathcal{D}$ be a deductive system containing only positive rules and the rule of substitution. Then, if $\mathcal{D}$ is C-complete for a unified logic $L$, the system $\mathcal{D}'$ obtained from $\mathcal{D}$ by postulating Co and Fu, is $L$-complete.

**Example**

One can take any calculus that defines the classical logic and contains the rule of substitution, and convert it to a C-complete deductive system by adding anti-axiom $\Theta p$. If we add to this deductive system Co and Fu, we obtain an $L$-complete system.
Moreover, if we take any calculus with the rule of substitution defining the classical logic, we can convert it into an \( \mathcal{L} \)-complete deductive system by adding the rules Co, Fu and \( r := \oplus p, \oplus \neg p/\triangleright \). The needed anti-axiom \( \ominus p \) is derivable:
Theorem

For any finite sets of statements $\Gamma, \Delta$ and any statement $\alpha$,  

\[
\begin{align*}
\frac{\Gamma, \alpha}{\Delta} & \quad \vdash S \quad \frac{\Gamma}{\Delta, \bar{\alpha}} \\
& \\
\frac{\Gamma}{\Delta, \alpha} & \quad \vdash S \quad \frac{\Gamma, \bar{\alpha}}{\Delta}
\end{align*}
\]

and
Theorem

For any finite sets of statements $\Gamma, \Delta$ and any statement $\alpha$,

\[
\frac{\Gamma, \alpha}{\Delta} \vdash S \quad \frac{\Gamma}{\Delta, \alpha} \vdash S \quad \frac{\Gamma}{\Delta, \alpha} \vdash S \quad \frac{\Gamma, \alpha}{\Delta}
\]

and

Corollary

Let $(\Gamma, R \cup S)$ be a deductive system defining a unified logic $L$. Then there is a system of positive rules $R^+$, such that $(\Gamma, R^+ \cup S)$ defines $L$. 
**Ł-complete system for the Classical Logic**

### Theorem

The deductive system consisting of the below rules\(^a\) is Ł-complete for the classical logic \(Cl\).

\[
\begin{array}{lll}
(i) & E_i = \frac{\oplus p, \oplus(p \to q)}{\oplus q} & li_1 = \frac{\oplus q}{\oplus(p \to q)} & li_2 = \frac{\oplus(p \to (q \to r))}{\oplus(p \to q), \oplus(p \to r)} \\
(c) & E_{cl} = \frac{\oplus p \land \oplus q}{\oplus p} & E_{cr} = \frac{\oplus p \land \oplus q}{\oplus q} & l_c = \frac{\oplus p, \oplus q}{\oplus(p \land q)} \\
(d) & E_{dl} = \frac{\ominus(p \lor q)}{\oplus p} & E_{dr} = \frac{\ominus(p \lor q)}{\oplus q} & l_d = \frac{\oplus(p \to r), \oplus(q \to r)}{\oplus((p \lor q) \to r)} \\
(n) & E_n = \frac{\oplus p, \ominus p}{\triangle} & l_n = \frac{\ominus}{\oplus p, \ominus p} \\
(r) & C = \frac{\oplus p, \ominus p}{\triangle} & F_u = \frac{\ominus}{\oplus p, \ominus p} & S_b = \frac{\oplus A}{\ominus \sigma(A)}
\end{array}
\]

---

\(^a\)The positive m-rules that define the positive part of \(Cl\) are similar to m-rules from Shoesmith and Smiley, Multiple-conclusion logic, 2008.

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Alex Citkin Deductive systems with unified multiple-conclusion rules
Final remarks

The rule \( \triangledown/ \oplus p, \ominus p \) (and not the \( \triangledown/ \oplus p, \ominus \neg p \), or \( \triangledown/ (p \lor \neg p) \)) expresses the Law of Excluded Middle. The Law of Excluded Middle is not about disjunction and negation: you may have it for the systems without disjunction and negation. The Law of Excluded Middle means that

One always can assert or reject any given proposition.
Final remarks

The rule $\nabla/ \oplus p, \ominus p$ (and not the $\nabla/ \oplus p, \ominus \neg p$, or $\nabla/ \oplus (p \lor \neg p)$) expresses the Law of Excluded Middle. The Law of Excluded Middle is not about disjunction and negation: you may have it for the systems without disjunction and negation. The Law of Excluded Middle means that

One always can assert or reject any given proposition.

Accordingly, the rule $\oplus p, \ominus p/\blacktriangle$ expresses the Law of Non-Contradiction, which is not about conjunction and negation; it means that

One cannot assert and reject the same proposition at the same time.
Thank you for your patience and attention.

Alex Citkin