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Remarks Concerning the Phenomenological Foundations of Mathematics

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In this paper I investigate the phenomenological approach to foundations of mathematics. Phenomenological reflection plays the certain role in extension of mathematical knowledge by clarification of meanings. The phenomenological technique pays our attention to our own acts in the use of the abstract concepts. Mathematical constructions must not be considered as passive objects, but as categories are given in theoretical acts, in categorical experiences and in our senses. Phenomenology moves like a category theory from formal components of knowledge to the dynamics of constitutive process.

Keywords: Phenomenology of mathematics, infinite structures, category theory, abstract objects, movement, visibility

1.

The first division of this article represents some reflections about the essential connection of transcendental phenomenology with formal proof theory. The main Edmund Husserl's scientific interests were philosophical questions of logic, mathematical problems and phenomenological foundations of mathematics.

Let us consider notion of definite multitude (definiten Mannigfaltigkeit). Multitude is an idea of form of infinite objective region, which manifests itself in the unity of nomological science. Such idea of a deductive science is equivalent to axiomatic definite system. The general feature of formalistic system is *completeness* (vollstandiges) because notion of multitude includes principal definable for all elements of this multiplicity. In given point Husserl draws together with David Hilbert's logical constructions, directed on realization of the program of the foundation of mathematics. Modern investigators discuss developments in mathematics emphasizing finitist methods, for example, proof mining. However "naturlich knupfen sich hieran hochst bedeutsame Probleme... wie kann man wissen, wie beweisen, ein Axiomensystem ein definites ist, ein 'vollstandiges'?" [4, p. 84] [Of course this problem is bound up with other important questions. . . how one can know and how to prove that an axiomatic system is definable and "complete"] (See also [5]). In other words, axiom of completeness paves the way for mathematical cognition in the same direction with logician thought, connected by the notion of *Definitheit*. And so first conclusion of this paper is dependence of husserlian phenomenology on the formalistic conceptions. Phenomenological description of eidetic variations is impossible to separate from formalistic constructions. The combination of intentional and formal structures is the basic condition for real possibility of sciences itself and philosophy as rigorous science.

On the other hand phenomenological analysis leaves definite traces on understanding of formal proofs. It is evident that any logical or mathematical proof differs from ordinary empirical notification. The logical sense of proof intends to an obvious perceiving of consequences, when we conclude from existence of a certain state of affair to other. Husserl insists on correspondence of subjective acts of proving to objective conclusions and proofs. In his works Husserl has not given enough attention for proof theory and the systematical accounts of this topic are absent. But he is clearly conscious of the need of formalistic proving treatment for program of reconstruction of philosophy as rigorous science. Generally, without application of formalistic procedures is impossible any science.

Proof theory as a part of formal logic may be considered in the two-sidedness direction: in the subjective and objective main themes. In *FTL* Husserl analyses mind's work from the point of view of objective achievements (geleisteten Ergebnisse) and of the same time from the point of view of subjective activity (leistenden Tatigkeiten und Habitualitaten). Objective logic includes all forms of judgments and hardened cognitive structures which represent practical results. On the basis of these results we may also build new structures, judgments or proofs. All these cognitive products have not only transient being, but also have objective validity of the sound value.

Unlike formal logic the transcendental logic investigates subjective forms of theoretic mind. Here we must solve the problem of living activity (in lebendigem Vollzug) of human mind. All proofs have origin in proving mind. Theoretical constructions and proofs are categorical objects in categorical experience which make sure the possibility of pure formalization. Proofs are not reduced to mental acts of proving (Beweisen). Some investigators follow the traces of eidetic method in mathematical intuition not only in material mathematics like in geometry, but also in formal mathematics. Eidetic variation serves as the source of intuitivity in proofs. Thus phenomenology provides us detailed description of what we are doing in proofs.

2.

And in fact, phenomenological influence on mathematics is considerably wider and in general some philosophical reflection plays the certain role in mathematical success. This conclusion belongs, as far as I know, to Kurt Gödel who in his unpublished lecture reflecting on the modern development of the foundations of mathematics, turns away from the formalism to the phenomenological procedures. In the lecture he had intended to describe the contemporary state of mathematics in terms of philosophical notions, directing attention mostly at the foundational program of Hilbert which represents mathematical theories by means of finitary reasoning. The serious theoretical mistake of this program is the refusal from philosophical reflection on the foundations of mathematics. It means that formalistic conception excludes the epistemological analysis from mathematics.

Meanwhile, the attempt to introduce a new discipline, metamathematics, as it turned out, was a manifestation of some inclination to empirical reflection, viz. reflection on the combinatorial properties of any concrete symbols. Furthermore the method of arithmetisation of metamathematics is also the peculiar modification of empirical point of view.

However Gödel says about the need to reflect on meanings of mathematical concepts. "Obviously, this means that the certainty of mathematics is to be secured not by...the manipulation of physical symbols — but rather by cultivating (deepening) knowledge of the abstract concepts" [2, p. 383]. The question is to search out the possibility of extension of mathematical knowledge by a clarification of meanings which consist in focusing on our own acts in the use of mathematical concepts. The phenomenological technique should make in us a new state of consciousness. In any case phenomenological position is sufficiently potent because it takes into consideration both philosophical meditations and mathematical results. Thus, phenomenological clarification of the meaning (or meanings) of primitive concepts should not be exclude from mathematical region.

The formalistic idea is to reduce our mathematical knowledge to concrete sign configurations and combinatorial operations on such symbols, but it is only part of mathematical work and it does not give us the exhaustive explanation of mathematical knowledge. Opposite, Gödel's incompleteness theorem in addition show that "in the transition from evidence to pure formalism something is lost... many mathematical statements express noemata that are not captured in purely syntactical terms" [10, p. 152]. According to Gödel a systematic method for clarification of meaning in giving definitions was produced in the phenomenology founded by Husserl. The phenomenological procedure pays our attention to our own acts in the use of the abstract concepts, giving us the distinctive foundation to mathematics by means of reflexive analysis of these mental acts. Phenomenological reflection of the constitution of mathematical objects on that ground must be realized in two directions: first one is the theory of meaning and second one is the theory of objects. One of them uses only categories of meaning — *Bedeutungskategorien*, and other puts into practice formal objective categories — *gegenstandliche Kategorien* (*FTL* §§ 27, 33, 34, 37, 42 etc).

3.

Then some mathematical theories involve the *infinite* structures. Despite of intuitionist efforts to banish the infinite from mathematics or reduce such considerations to a game of symbols, we cannot eliminate mathematical concepts which base themselves upon the reality of the infinite.

In this connection appeared the appropriate question: how can the infinite structures possibly be grasping by human mind? "What are actually present in consciousness are finite structures. How can they give us access to mathematical structures or forms that are in representative cases infinite? Cannot Husserl's own manifolds be infinite?" [3, p. 99]. The problem does not seem to me so complicated after all, but very important for mathematical progress. In this question the phenomenological description accentuate on subjective (intentional) topics of a pure analytic. The basic form 'and so on' or iterative infinity has the subjective correlate which is represented as 'may be added anew'. Such expression is none other than a mathematical idealization; however it plays the sense putting role. Iterative form buildings, for example $a + 1$, are not a game with empty thoughts, on the contrary the similar constructions are fitted for cognition of things.

Mathematics is realm of infinite constructions, a kingdom of ideal existences which are considered not only in finite sense, but also as the constructive infinites (konstruktiven Unendlichkeiten). In this point Husserl had run once more into problem of subjective constitution which must be a new method of infinite constructions, "der Methode, in der das 'und so weiter' verschiedenen Sinnes und die Unendlichkeiten als neuartige kategoriale Gebilde... evident warden" [4, p. 167] [The method which makes obvious senses of ‘and so on’ and infinities as novel categorical structures.]. In practice, mathematicians applying to objective forms of categories, namely to diagrams, manifolds, quadratic forms etc., forget as a rule that any categories proceed in the categorical acts of human mind by drawing a line (an arrow), by putting in order and also in acts of combining and calculation. The varieties of mental acts in mathematics make up the *real “dark”* side of its ideal objects.

In some contemporary philosophical investigations authors try to arouse interest for correlation phenomenology and mathematics, whereas this question may be formulated otherwise: what mathematical issues outline the phenomenological fields of knowledge, what kinds of mathematical search are favors the development of phenomenology? Thus the aim of next divisions is to attach importance of phenomenological reflections in mathematical investigations, in particular for the higher category theory, quadratic forms, number theory, integration of measures, theory of scale [8, p. 392].

4.

Phenomenology is able to make simultaneously a careful close study of classical and constructive mathematics in substantial correlation. The initial problem as before very actual can be formulated in the next thesis: reflective profound study of intentional origins of fundamental mathematical concepts, such as objects, morphisms, manifolds, categories and so on is the common aim both for philosophers and for mathematicians. It is a matter of common concern. Mathematical constructions must be considered, according to phenomenology, not as passive objects, but rather as categories given in theoretical acts [kategoriale Erfahrung].

The task of justifying phenomenologically the symbolic character of mathematics, in particular the universal language of higher category theory, I will denominate as phenomenological foundations of mathematics — PhFOM. Any solution in present case necessarily will demand carrying out of a subjective-guided investigation.

When we admiringly prove mathematical theorems, we always neglect origin and lineage of all these perfections. Fortunately, not every mathematicians forget the primary source of their imaginary constructions, I mean first and foremost Descartes, Leibniz, Bolzano, Frege, Poincare, Hadamard, Bourbaki, Gödel and others. Mathematician, who makes up his mind to grasp the foundations of mathematics, is a philosopher, because he transcend limits of his science. The working mathematician is obliged to take into account the intellectual changeableness. The purpose of these phenomenological investigation is to describe the objects of mathematics in their correlation with intellectual activity of our mind. Phenomenological descriptions are suited to the concepts and methods of geometry, topology, algebra, but most of all to the modern theory of categories.

5.

For adaptation of category-theoretic ideas mathematicians introduce the more suitable formalism of ∞-categories (J. Lurie) or quasi-categories (A. Joyal). They establish a vocabulary which contains categorical analogues of the concepts from ordinary category theory. A category C consists of the following components:

(i) objects of $C(X, Y, Z)$

(ii) morphisms from X to Y $(f : X \to Y)$

(iii) an identity morphism Hom C (X, X)

(iv) composition map for every triple of objects

(v) composition functors which are required to satisfy associative law.

This terminology may be applied to ∞-categories (topological or simplicial categories). It makes sense to speak of finitely presented ∞-category. Such mathematical structures like ordinary categories can be described by generators and relations. For example, higher category generated by a single morphism $g : Y \to Y$ is a finitely presented. But higher category theory gives raise the technical difficulties inherent in working with too large objects. Considering presentable and accessible ∞-categories, Lurie introduce the notion of a "locally small" ∞-category, which has small morphism spaces for any fixed pair of objects. "The theory of accessible ∞-categories is a tool which allows us to manipulate large ∞-categories as if they were small without fear of encountering any set-theoretic paradoxes" [6, p. 415]. He begins with the most intuitive approach to the formalism of ∞-categories using topological categories up to a weak homotopy equivalence of topological spaces.

Further, in contemporary mathematical investigations have arisen discussions about the question: what advantages has categorical philosophy of mathematics over set-theoretic foundations? One of the most advantages of category theory is a good universal language with a sufficient level of generality. This language is closer to the mathematical content and categorical structures are not lost in translation. Moreover, set-theoretic foundations concern the binary relation of membership, while theory of categories axiomatizes the ternary relation of composition and applies quadruple diagrams. The difference is that Zermelo-Fraenkel axioms (ZFC) take membership while Elementary Theory of the Category of Set (ETCS) takes composition of functions. "So ETCS is closer to the working methods of mainstream mathematics than ZFC is, and like those methods it rests ontologically on form and structure rather than membership and substance" [7, p. 151].

Some investigators try to bring phenomenology in correspondence with category theory, for there are many points of contiguity between categorical foundations and phenomenological descriptions. The goals of category theory are not completely mathematical, but also they outline epistemological perspectives providing ample opportunities to apply phenomenological approach in mathematics.

6.

The phenomenological epistemology of mathematical experience in category theory is permissible by following considerations. No object could be conceivable without a sense horizon. Of course mathematical objects are included in conscious horizon of mathematician. "En d'autres termes, c'est dans la correlaton cogito-cogitatum que prend racine la pensée mathematique. . . , la structure interne des categories reflete en partie la dichotomie noeme/noese" [9, p. 416] [In other words, the question is that cogito-cogitatum correlation had taken root in mathematical reflection. . . , inner structure of categories has a partial effect on dichotomy noema/noesis]. Any objectifying act consists of some particular intentions, for example when I am proving a theorem or when I am trying to present properties of mathematical constructions by means of arrows and diagrams. Such act lasts during the certain period of time and all particular intentions are combined in synthesis of this lengthy act.

But on the other hand, the modern category theory naturally leads to renewal of phenomenology of mathematics. I hold that category theory favor the development of phenomenology and it can to create a new approach to classical phenomenological notions and distinctions. Category theory has introduced in formal undertaking one decisive correction offering to consider mathematical structures in movement as opposed to habitual practice. Thus, one may establish a fact of theoretical transformation, namely when categorist overstep the limits of structuralistic intuition.

The question is not merely operate on static objects, quite the contrary, mathematical structures become dynamical objects provided with arrows. The chief phenomenological interest has moved like category theory from formal components of knowledge to the dynamics of constitutive process.

7.

In the category theory mathematicians look for representation their abstract entities appealing to our senses, in this case to visibility of movement. However these remarks are essential not only to category theory, but also to other mathematical fields of knowledge. John Conway raises quite rightful question: can we see the "values" of quadratic forms? In his lectures devoted to the sensual quadratic forms he presents a visual method to display the values by changing viewpoint. As a result "theorems that once had to be proved algebraically or arithmetically can now become so obvious that they no longer require proof" [1, p. 25]. Further he considers concept of audibility of a lattice: audible properties of a lattice are determined by θ-function. In addition he recovers the structure of rational forms according to the primary fragrances and gives us the opportunity to have a sensation of a taste of number theory.

It has become apparent upon phenomenological analysis that it is wrong when we remove subjective (or egological) structures from mathematical investigations, for actually abstract mathematical notions, for example, the space of geometric points is not indifferent to the agent of demonstration. This analysis may be also represented in terms of genetic method for some mathematical theories. And finally, necessity of subjective-orientated investigations is conditioned by scientific tasks of measuring. As a matter of fact measuring is not a mechanical applying standard measure to outer appearances, but outstripping modeling of reality.

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