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Символическая логика
Symbolic Logic

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On First-order Theories Which Can Be Represented by Definitions

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In the paper we consider the classical logicism program restricted to first-order logic. The main result of this paper is the proof of the theorem, which contains the necessary and sufficient conditions for a mathematical theory to be reducible to logic. Those and only those theories, which don't impose restrictions on the size of their domains, can be reduced to pure logic.

Among such theories we can mention the elementary theory of groups, the theory of combinators (combinatory logic), the elementary theory of topoi and many others.

It is interesting to note that the initial formulation of the problem of reduction of mathematics to logic is principally insoluble. As we know all theorems of logic are true in the models with any number of elements. At the same time, many mathematical theories impose restrictions on size of their models. For example, all models of arithmetic have an infinite number of elements. If arithmetic was reducible to logic, it would had finite models, including an one-element model. But this is impossible in view of the axiom $0 \neq x'$.

Keywords: definition, definability, predicate calculus, theory, logicism

1. Logicism

As we know the main idea of logicism was that mathematics was an extension of logic and was reducible to logic by appropriate definitions.

One of the explications of logicism might look like if you are given a theory T with the set of postulates Ax. It is required to find such a set of logical definitions DF of mathematical notions of the theory T that for every formula $B \in L_T$ holds:

$$Ax \vdash B \Leftrightarrow DF \vdash B.$$

As we know the attempt to implement the program of classical logicism has failed. It needs the higher-order logic, and far from intuitively obvious axioms: reducibility, multiplicativity (choice) and infinity, which can hardly be called logical. This was a major rebuke to the logicism.

It is interesting to find an answer to the more specific question:

To what limits classical logicism program can be implemented in the first-order predicate logic?

2. Defining new predicate symbols

We assume that the language of first-order predicate calculus is defined in the standard way as the set of terms and formulas over the signature Σ, which consists of nonlogical relational and functional symbols. We write $L(\Sigma)$ for the first-order language over signature Σ. Models are pairs $M = \langle D, I\rangle$, where D is a non-empty set of individuals, and I is an interpretation of the function and predicate symbols in the domain D. The relations *"formula A is true in the model M for value assignment to individual variables g"* and *"formula A is true in the model M"* are defined as usual and are written as $M, g \vDash A$ and $M \vDash A$.

A first-order theory in the language $L(\Sigma)$ is a set of logical axioms and non-logical postulates closed by derivability. Predicate calculus is the first-order theory with the empty set of non-logical postulates. We consider equality axioms as non-logical postulates.

We can extend the language of a theory by definitions of new predicate symbols, which have the following form:

$$\forall x_1 \dots x_n (P(x_1, \dots, x_n) \equiv A).$$

The definition must satisfy the conditions:

1. $P \notin \Sigma$.
2. $A \in L(\Sigma)$.
3. The variables $x_1, \dots, x_n$ are pairwise distinct.
4. The set of free variables of A is included in $\{x_1, \dots, x_n\}$.

The newly defined predicate symbol P must be added to the signature Σ. As the result, there is a transition from the language $L(\Sigma)$ to the language $L(\Sigma \cup \{P\})$.

In the language of the first order predicate calculus, we can define the universal n-ary predicate U^n by the following definition:

(DU) $\quad \forall x_1 \ldots x_n(U^n x_1, \ldots, x_n \equiv Px_1 \vee \neg Px_1).$

The definition allows us to prove $DU \vdash \forall x_1 \ldots x_n Ux_1, \ldots, x_n$.

This example is interesting because in the right part of the definition we use an arbitrary predicate symbol of the signature of the first order predicate calculus, but with the help of it, we define the specific predicate symbol with the specific properties.

As another example, we can give a definition of a symmetric relation. Let B be an arbitrary predicate symbol of the signature. We accept the following definition:

(DS_1) $\quad \forall xy(S_1 xy \equiv \forall uv(Buv \supset Bvu) \supset Bxy)$

Let us show that $DS_1 \vdash \forall xy(S_1 xy \supset S_1 yx)$.

1. $S_1 xy$ - hyp
2. $\forall uv(Buv \supset Bvu) \supset Bxy$ - from 1, DS_1 by replacement
3. $\forall uv(Buv \supset Bvu)$ - hyp
4. Bxy - from 2, 3 by *m.p.*
5. $Bxy \supset Byx$ - from 3 by $\forall_{el}$
6. Byx - from 4, 5 by *m.p.*
7. $\forall uv(Buv \supset Bvu) \supset Byx$ - from 3-6 by $\supset_{in}$
8. $S_1 yx$ - from 7, DS_1 by replacement
9. $S_1 xy \supset S_1 yx$ - from 1-8 by $\supset_{in}$

There is another way to define a symmetric relation:

(DS_2) $\quad S_2 xy \equiv \forall uv(Buv \supset Bvu) \& Bxy.$

Let us show that $DS_2 \vdash \forall xy(S_2 xy \supset S_2 yx)$.

1. $S_2 xy$ - hyp
2. $\forall uv(Buv \supset Bvu) \& Bxy$ - from 1, DS_2 by replacement
3. $\forall uv(Buv \supset Bvu)$ - from 2 by $\&_{el}$
4. Bxy - from 2 by $\&_{el}$
5. $Bxy \supset Byx$ - from 3 by $\forall_{el}$
6. Byx - from 4, 5 by *m.p.*
7. $\forall uv(Buv \supset Bvu) \& Byx$ - from 3, 6 by $\&_{in}$
8. $S_2 yx$ - from 7, DS_2 by replacement
9. $S_2 xy \supset S_2 yx$ - from 1-8 by $\supset_{in}$

These examples motivate us to find the general criterion of definability of the specific predicates with the help of predicate logic.

DEFINITION 1. The first-order theory T in a language $L(\Sigma)$ with finite set of non-logical axioms Ax is *definitionally embeddable* into predicate calculus if and only if there are a signature Σ' and a set of definitions DT of symbols $\Sigma \setminus \Sigma'$ by formulas of $L(\Sigma')$ which met the following condition:

$$\text{If } B \in L(\Sigma), \text{ then } Ax \vdash B \Leftrightarrow DT \vdash B.$$

This definition is some variant of the notion of definitional embeddability of theories, which was proposed by V.A. Smirnov in [2], [3, p. 65].

3. Auxiliary lemmas

To formulate the main theorem, we need to define function π, which translates formulas of first-order theories into formulas of the propositional logic. This function simply "erases" all terms and quantifiers in formulas.

DEFINITION 2.

1. $\pi(P(t_1, \ldots, t_n)) = P$.
2. $\pi(\neg A) = \neg\pi(A)$.
3. $\pi(A \bigtriangledown B) = \pi(A) \bigtriangledown \pi(B)$, where $\bigtriangledown \in \{\&, \vee, \supset, \equiv\}$.
4. $\pi(\Sigma x A) = \pi(A)$, where $\Sigma \in \{\forall, \exists\}$.

LEMMA 1. Let v be some truth-value assignment to propositional variables that is in the standard way extended to all formulas of propositional logic, then the next statements are true:

(A) If for each atomic subformula $P_i(\vec{t})$ of formula A holds $\dot{\forall} g[M, g \models P_i(\vec{t}) \Leftrightarrow v(\pi(P_i)) = True]$, then it holds $\dot{\forall} g[M, g \models A \Leftrightarrow v(\pi(A)) = True]$.

(B) If for each atomic subformula $P_i(\vec{t})$ of formula A holds $\dot{\forall} g[M, g \models P_i(\vec{t}) \Leftrightarrow v(\pi(P_i)) = True]$, then it holds $[M \models A \Leftrightarrow v(\pi(A)) = True]$.

Proof.

(A) We prove the statement by structural induction. The basis of induction is the condition of the lemma $\dot\forall g[M, g \models P_i(\vec{t}) \Leftrightarrow v(\pi(P_i)) = True]$. So we have to prove the induction step.

Case 1. $A = \neg B$

1. $M, g \models \neg B$ - hyp
2. $\dot\forall h[M, h \models B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $M, g \not\models B$ - from 1 by definition
4. $M, g \models B \Leftrightarrow v(\pi(B)) = True$ - from 2
5. $v(\pi(B)) = False$ - from 3, 4 and definition v
6. $v(\neg\pi(B)) = True$ - from 5 by definition v
7. $v(\pi(\neg B)) = True$ - from 6 by definition π

1. $v(\pi(\neg B)) = True$ - hyp
2. $\dot\forall h[M, h \models B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $v(\neg\pi(B)) = True$ - from 1 by definition π
4. $v(\pi(B)) = False$ - from 3 by definition v
5. $M, g \models B \Leftrightarrow v(\pi(B)) = True$ - from 2
6. $M, g \not\models B$ - from 4, 5
7. $M, g \models \neg B$ - from 6 by definition

Case 2. $A = B\&C$

1. $M, g \models B\&C$ - hyp
2. $\dot\forall h[M, h \models B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $\dot\forall h[M, h \models C \Leftrightarrow v(\pi(C)) = True]$ - inductive hyp
4. $M, g \models B$ - from 1 by definition
5. $M, g \models C$ - from 1 by definition
6. $M, g \models B \Leftrightarrow v(\pi(B)) = True$ - from 2
7. $M, g \models C \Leftrightarrow v(\pi(C)) = True$ - from 3
8. $v(\pi(B)) = True$ - from 4, 6
9. $v(\pi(C)) = True$ - from 5, 7
10. $v(\pi(B)\&\pi(C)) = True$ - from 8, 9 by definition v
11. $v(\pi(B\&C)) = True$ - from 10 by definition π

1. $v(\pi(B\&C)) = True$ - hyp
2. $\dot\forall h[M, h \models B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $\dot\forall h[M, h \models C \Leftrightarrow v(\pi(C)) = True]$ - inductive hyp

4. $v(\pi(B)\&\pi(C)) = True$ - from 1 by definition π
5. $v(\pi(B)) = True$ - from 4 by definition v
6. $v(\pi(C)) = True$ - from 4 by definition v
7. $M, g \vDash B \Leftrightarrow v(\pi(B)) = True$ - from 2
8. $M, g \vDash C \Leftrightarrow v(\pi(C)) = True$ - from 3
9. $M, g \vDash B$ - from 5, 7
10. $M, g \vDash C$ - from 6, 8
11. $M, g \vDash B\&C$ - from 9, 10 by definition

Case 3. $A = \forall xB$

1. $M, g \vDash \forall xB$ - hyp
2. $\dot{\forall} h[M, h \vDash B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $M, g' \vDash B$ - from 1 for arbitrary $g' \approx_x g$
4. $M, g' \vDash B \Leftrightarrow v(\pi(B)) = True$ - from 2
5. $v(\pi(B)) = True$ - from 3, 4
6. $v(\pi(\forall xB)) = True$ - from 5 by definition π

1. $v(\pi(\forall xB)) = True$ - hyp
2. $\dot{\forall} h[M, h \vDash B \Leftrightarrow v(\pi(B)) = True]$ - inductive hyp
3. $v(\pi(B)) = True$ - from 1 by definition π
4. $M, g \nvDash \forall xB$ - hyp
5. $M, g' \nvDash B$ - from 4 for some $g' \approx_x g$
6. $M, g' \vDash B \Leftrightarrow v(\pi(B)) = True$ - from 2
7. $M, g' \vDash B$ - from 3, 6
8. contradiction - 5, 7
9. $M, g \vDash \forall xB$ - from 4, 8

Since all logical connectives and the existential quantifier are definable through $\{\neg, \&, \forall\}$, the part **(A)** of the lemma is proved.

(B) The metalanguage statement $\dot{\forall} g[M, g \vDash A \Leftrightarrow v(\pi(A)) = True]$ implies the statement $[\dot{\forall} g(M, g \vDash A) \Leftrightarrow v(\pi(A)) = True]$, but $\dot{\forall} g(M, g \vDash A)$, it means the same as $M \vDash A$. So part **(B)** of the lemma follows trivially from the part **(A)**.

□

If Ax is the set of formulas then $\pi(Ax)$ will denote the set of formulas $\{\pi(A) | A \in Ax\}$.

Lemma 2. If T is a theory with a set of axioms Ax then the set of formulas $\pi(Ax)$ is consistent if and only if for every set D there exists such a function of interpretation I, that $M = \langle D, I\rangle$ and for each $A \in Ax$ holds $M \vDash A$.

Proof.

($\Rightarrow$) Suppose, $\pi(Ax)$ is consistent. It follows that there is the truth-value assignment v to propositional variables, at which all the formulas $\pi(Ax)$ are true.

Suppose that D is a non-empty set of individuals. We define the function of interpretation I of nonlogical language symbols in the set D. Let us choose an element e of the set D.

(1) If c — individual constant then $I(c) = e$.

(2) If f is n-ary function symbol then $I(f) : D \times \ldots^n \times D \to \{e\}$.

(3) For any n-ary predicate symbol P_i, if $v(\pi(P_i)) = True$ then $I(P_i) = D \times \ldots^n \times D$, else $I(P_i) = \emptyset$.

Let us show that in the model $M = \langle D, I\rangle$ holds $M \vDash Ax$.

According to the constructed model, $\dot{\forall} g[M, g \vDash P_i(t) \Leftrightarrow v(\pi(P_i))]$. From the Lemma 1 we obtain $M \vDash A \Leftrightarrow v(\pi(A))$. Because for all $A \in Ax$ holds $v(\pi(A)) = True$, so we have $M \vDash A$.

($\Leftarrow$) The proof is trivial, since the consistency of $\pi(Ax)$ follows from the existence of a one-element model $M = \langle \{a\}, I\rangle$. □

4. The main theorem

The following theorem is a stronger form of the theorem proved in [1].

Theorem 1. *Let T be a first-order theory in a language $L(\Sigma)$ with a finite set of closed non-logical postulates $Ax = \{A_1, \ldots, A_k\}$.*

(A) *T is definitionally embeddable into the first-order predicate calculus if and only if the set of formulas $\{\pi(A_1), \ldots, \pi(A_k)\}$ is logically consistent.*

(B) *T is definitionally embeddable into the first-order predicate calculus if and only if it does not impose any restrictions on the power of models.*

Proof.

(A) ($\Leftarrow$) We must prove that if the set of formulas $\{\pi(A_1), \ldots, \pi(A_k)\}$ is logically consistent then the theory T is definitionally embeddable into the first order predicate calculus.

Let $\{P_1, \ldots, P_m\}$ be the set of all predicate symbols of signature Σ, which occur in nonlogical postulates $\{A_1, \ldots, A_k\}$.

The logical consistency of $\{\pi(A_1), \ldots, \pi(A_k)\}$ means that there exists at least one truth-value assignment v to propositional letters $\pi(P_1), \ldots, \pi(P_m)$ with property $v(\pi(P_1)) = True, \ldots, v(\pi(P_m)) = True$. Let us fix some such assignment v.

Take the signature Σ' which satisfes the two conditions:

- $\Sigma \setminus \Sigma' = \{P_1, \ldots, P_m\}$.
- For each predicate symbol $P_i \in \{P_1, \ldots, P_m\}$ there exists such a predicate symbol R_i of the corresponding arity, that $R_i \in \Sigma'$.

We use $\widehat{Ax}$ to denote the conjunction $A_1 \& \ldots \& A_k$ of all postulates $A_1, \ldots, A_k$ and $\widehat{Ax}\,[R/P]$ to denote the result of renaming all occurrences of symbols $P_1, \ldots, P_m$ into $R_1, \ldots, R_m$.

We associate the definition with each predicate symbol $P_i \in \{P_1, \ldots, P_m\}$ by the following rule:

1) If $v(\pi(P_i)) = True$, then

$$\forall \vec{x}(P_i(\vec{x}) \equiv \widehat{Ax}\,[R/P] \supset R_i(\vec{x}))$$

2) If $v(\pi(P_i)) = False$, then

$$\forall \vec{x}(P_i(\vec{x}) \equiv \widehat{Ax}\,[R/P]\,\& R_i(\vec{x}))$$

Let $DT = \{D_1, \ldots, D_m\}$ be the set of all definitions.

(A.1) We must show that if $B \in L(\Sigma)$ and $Ax \vdash B$, then $DT \vdash B$. By the properties of the deducibility relation it suffices to show $DT \vdash \widehat{Ax}$. By the completeness theorem of the first-order predicate calculus it is equivalent to $DT \vDash \widehat{Ax}$.

Let $M = \langle D, I \rangle$ be a model in which all formulas of DT are true.

Since the formula $\widehat{Ax}\,[R/P]$ is closed we have either $M \vDash \widehat{Ax}\,[R/P]$ or $M \vDash \neg\widehat{Ax}\,[R/P]$.

Case 1. $M \vDash \widehat{Ax}\,[R/P]$. For each P_i we have one of the following two subcases:

Subcase 1.1. $v(\pi(P_i)) = True$

$M, g \vDash P_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P] \supset R_i(\vec{t}) \Leftrightarrow$

$M, g \vDash R_i(\vec{t})$

Subcase 1.2. $v(\pi(P_i)) = False$

$M, g \vDash P_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P]\,\&R_i(\vec{t}) \Leftrightarrow$
$M, g \vDash R_i(\vec{t})$

In each case P_i is interpreted as R_i and therefore $M \vDash \widehat{Ax}$.

Case 2. $M \vDash \neg\widehat{Ax}\,[R/P]$. For each P_i we have one of the following two subcases:

Subcase 2.1. $v(\pi(Pi)) = True$

$M, g \vDash P_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P]\,\&R_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P]\,\&R_i(\vec{t}) \vee \neg\widehat{Ax}\,[R/P]\,\&(R_i(\vec{t}) \vee \neg R_i(\vec{t})) \Leftrightarrow$
$M, g \vDash \neg\widehat{Ax}\,[R/P]\,\&(R_i(\vec{t}) \vee \neg R_i(\vec{t})) \Leftrightarrow$
$M, g \vDash R_i(\vec{t}) \vee \neg R_i(\vec{t}) \Leftrightarrow$
$v(\pi(Pi))$

Subcase 2.2. $v(\pi(Pi)) = False$

$M, g \vDash P_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P]\,\&R_i(\vec{t}) \Leftrightarrow$
$M, g \vDash \widehat{Ax}\,[R/P]\,\&R_i(\vec{t}) \vee \neg\widehat{Ax}\,[R/P]\,\&(R_i(\vec{t})\&\neg R_i(\vec{t})) \Leftrightarrow$
$M, g \vDash \neg\widehat{Ax}\,[R/P]\,\&(R_i(\vec{t})\&\neg R_i(\vec{t})) \Leftrightarrow$
$M, g \vDash R_i(\vec{t})\&\neg R_i(\vec{t}) \Leftrightarrow$
$v(\pi(Pi))$

For all atomic formulas $P_i(\vec{t})$ and all assignments g to individual variables we have $M, g \vDash P_i(\vec{t}) \Leftrightarrow v(\pi(P_i))$. The value of the atomic formula $P_i(\vec{t})$ doesn't depend on the particular assignments of values to individual variables. As a result, according to Lemma 1, we obtain $M \vDash \widehat{Ax} \Leftrightarrow v(\pi(\widehat{Ax}))$. But according to the properties of the function v it holds $v(\pi(A_1)) = True, \ldots, v(\pi(A_k)) = True$, and $\widehat{Ax}$ is the conjunction of $A_1, \ldots, A_k$. Hence $v(\pi(\widehat{Ax})) = True$ and $M \vDash \widehat{Ax}$.
With the help of the completeness theorem of the first-order predicate calculus, we obtain $DT \vdash \widehat{Ax}$.

(A.2) We must show that if $B \in L(\Sigma)$ and $DT \vdash B$, then $Ax \vdash B$. By the completeness theorem of the first-order predicate calculus it is equivalent to show that if $DT \vDash B$, then $Ax \vDash B$.

Let us assume that $B \in L(\Sigma)$ and $DT \vDash B$ but $Ax \nvDash B$. Then there exists such a model $M = \langle D, I\rangle$ of the theory T that $M \vDash \widehat{Ax}$ and $M \nvDash B$.

We can extend the model $M = \langle D, I\rangle$ to the model $M' = \langle D, I'\rangle$ in which all the formulas of DT will be true. It is sufficient to expand the domain of the function I so that the new function of interpretation I' ascribed value $I'(R_i) = I(P_i)$ to a predicate symbol R_i, and for all other functional and predicate symbols retained the same values as I.

Since $M \vDash \widehat{Ax}$, then in the model $M' = \langle D, I'\rangle$ by definition of I' we will have $M' \vDash \widehat{Ax}\,[R/P]$, and hence, $M' \vDash P_i(\vec{x}) \equiv \widehat{Ax}\,[R/P]\,\&R_i((\vec{x}))$ for each R_i. It follows that all the formulas DT are true in the model M'. Therefore by our assumption $DT \vDash B$ it must be $M' \vDash B$. However, the formula B doesn't contain symbols $R_1, \ldots, R_m$, while all the other descriptive symbols are interpreted in the same way as in the model M, and by assumption it must be $M', g \nvDash B$. We have obtained a contradiction. Therefore, the assumption that $Ax \vDash B$ does not hold is false.

(A) ($\Rightarrow$) We must prove that if a theory T is definitionally embeddable into first-order predicate calculus, then the set of formulas $\{\pi(A_1), \ldots, \pi(A_k)\}$ is consistent.

Let us assume that $Ax \vdash B \Leftrightarrow DT \vdash B$.

Take an arbitrary one-element model $M = \langle \{a\}, I\rangle$ for signature Σ'. For each predicate symbol $P_i \in \Sigma \setminus \Sigma'$, if it was introduced by definition $P_i(x_1, \ldots, x_n) \equiv D$, we expand the domain of the interpretation function I as follows:

$$I'(P_i) = \{\langle g(x_1), \ldots, g(x_n)\rangle : M, g \vDash D\}.$$

Note that since the domain of individuals consists of only one element, the function assigning values to individual variables, too, is the only one, and, consequently, predicate symbol P_i will be interpreted as either empty set $\emptyset$, or singleton $\{\langle a, \ldots, a\rangle\}$.

Performing this operation with all the new predicate symbols, we obtain the model $M' = \langle \{a\}, I'\rangle$, in which all the definitions of the set DT will be true.

Since we assumed that $Ax \vdash B \Leftrightarrow DT \vdash B$, then every axiom $A_i \in \{A_1, \ldots, A_k\}$ is derivable from DT. With the help of the completeness theorem of first-order predicate calculus, we obtain $DT \vDash A_i$. It means that there is at least one one-element model of the theory T, and hence, the set $\{\pi(A_1), \ldots, \pi(A_k)\}$ is logically consistent.

(B) The second part of the theorem follows from the part **(A)** and Lemma 2. □

5. Conclusion

The main theorem of this article can be considered as a solution of the classical logicism program for first-order theories. Those and only those theories which don't impose any restrictions on the power of their models can be reduced to pure logic.

Among of such theories we can mention the elementary theory of groups, the theory of combinators (combinatory logic), the elementary theory of topoi and many others.

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