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Game Theoretical Semantic for Relevant Logic[1]

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In 1979 D.E. Over proposed game theoretical semantics for first-degree entailment formulated by Anderson and Belnap. In order to extend this approach to include other systems of relevant logc (e.g., $\boldsymbol{R}$) we have two promoting facts. Firstly, there is Routley-Meyer's situational semantic for system $\boldsymbol{R}$ of relevant logic. Secondly, this semantics shows some resemblance with Wójcicki's situational semantic of non-fregean logic for which the situational game semantics was developed by author exploiting essentially the notion of non-fregean games. In the paper an attempt is done to give a partial account of these results and some conception of situational games developed which laid down into foundation of the game theoretical semantics of relevant logic $\boldsymbol{R}$.

Keywords: situational semantics, relevant logic, dialogue games, situational games

1. Introduction: situational games and relevant logic

In 1979 D.E. Over introduced game theoretical semantics for first-degree entailments (i.e. sentences of the form $(\varphi \to \psi)$ in which neither φ nor ψ contains $\to$) formulated by Anderson and Belnap [4]. According to Over there are to be two players in a game, White and Black, who make assertions according to the following rules:

($R1$) If a player asserts $(\varphi\wedge\psi)$, then he must assert φ or ψ at his opponent's choice.

($R2$) If a player asserts $(\varphi \vee \psi)$, then he must assert φ or ψ at his own choice.

($R3$) If a player asserts $\neg(\varphi \wedge \psi)$, then he must assert $(\neg\varphi \vee \neg\psi)$.

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($R4$) If a player asserts $\neg(\varphi \vee \psi)$, then he must assert $(\neg\varphi \wedge \neg\psi)$.

($R5$) If a player asserts $\neg\neg\varphi$, then he must assert φ.

The following rule for tautological entailment should be also added to ($R1$) – ($R5$):

($R6$) If a player asserts $(\varphi \to \psi)$, and another player asserts φ, then the first player must assert ψ.

Troubles with extending of Over's approach (as well as the other writer's approaches) to another relevance systems are connected with the lack of idea of the common game rule for relevant implication in that semantics (differs from the case of tautological entailment).

Yet apparently there are two directions of solving this problem which deserves to be inspected. Firstly, there is well-known Routley-Meyer's type situational semantic for system $\boldsymbol{R}$ of relevance logic and this semantics have much in common with Wojcicki's situational semantics of non-fregean logic. On the base of non-fregean situational semantics the situational game semantics was proposed [5] which essentially exploits the notion of so-called non-fregean games. So, attempting to find a partial account of these circumstances we can try to develope some conception of situational relevant games which would be laid down into foundation of the hypothetical game theoretical semantics of relevance logic.

Secondly, for Dishkant modal quantum logic [1] relational ternary semantics was introduced which partially (in bivalent case) coincides with the semantics for $\boldsymbol{R}$ and on this basis a game theoretical semantics was developed [6]. This also prompts us to relay on yielding game theoretical semantic for logic $\boldsymbol{R}$.

Some resemblance of approaches proposed in both cases can be induced by the pure logical reasons. As it well-known, according to Anderson-Belnap's ZDF theorem [2, p. 30] the zero-degree formulas (those containing only the connectives $\wedge$, $\vee$, $\neg$) provable in $\boldsymbol{R}$ (or $\boldsymbol{E}$) are precisely the theorems of classical logic. But non-fregean logic also contains calssical logic being its conservative extension.

Furthemore, since there is a situational semantics for $\boldsymbol{R}$ then a connective of relevant implication is determined by the conditions on the situations. But the same is true for the connective of so-called 'referential involvement' $\Rightarrow$ which could be construed as non-fregean situational implication (cf. [5]). And then the counterpart of the axiom of self-implication $A \to A$ is given by the non-fregean axiom $A \Rightarrow A$.

The same way the replacement theorem in $\boldsymbol{R}$ [2, p.26] $(A \leftrightarrow B) \wedge t \rightarrow (\chi(A) \leftrightarrow \chi(B))$ where $\chi(A)$ is any formula with perhaps some occurrences of A and $\chi(B)$ is the result of perhapls replacing one or more of those occurrences by B[2] is similar with non-fregean axiom $(A \Rightarrow B) \wedge t \rightarrow (\varphi(A) \Rightarrow \varphi(B))$.

Such similarity does not warrants complete resemblance of $\boldsymbol{R}$ and non-fregean logic and, strictly saying, it does not exist indeed. Distinction and resemblance could be assessed rather on the level of situational semantics which we have in our disposal for both systems.

2. Situational semantics for $\boldsymbol{R}$

The conception of a situation for the system $\boldsymbol{R}$ is given by means of the notion of *state of affairs*. A state of affairs (SOA) is a fact-like entity and it is what makes statements true at situations. From Edwin Mares' point of view from whose book [3, p. 61] we borrowed this definition SOA is just a sequence of the form:

$$\langle P, e_1, \dots , e_n \rangle$$

where P is a relation of type $(\tau_1, ..., \tau_n)$ and each e_i is an entity of the corresponding type τ_i. A sequence is just a set theoretic construct — it is an ordered set. SOA represents features or facts about worlds. But to do it accurately we need rather change such a definition:

A $SOA, < P, e_1, ..., e_n >$ accurately represents a fact at a world w if and only if $P(e_1, ..., e_n) \in w$.

This definition does not assume that in addition to worlds there are atomic facts. It might be that worlds themselves are each one big fact. Subworld facts do not play any role in this theory.

As will readily be observed, states of affairs are similar to the atomic situations in non-fregean logic. A set of states of affairs, as in case of the set of situations for non-fregean logic, can be a non-well-founded set with all its merits and demerits. But Mares maintains that his further framing might manage without such sophistication being asccomplished in the framework of standard set theory.

A situation is an ordered pair $s = (SOA(s), R(s))$. $SOA(s)$ is a set of states of affairs; these are the states of affairs that obtain at s. $R(s)$ is a set of ordered pairs. (t, u) is in $R(s)$ if and only if $Rstu$ holds in our frame. Thus, the set of situations itself determines which situations are related

[2] One clear role of the conjoined t is to imply $\chi \rightarrow \chi$ when χ (or $\chi(A)$) contains no occurrences of A or does but none of them is replaced by B.

to which in the frame. Mares supposes that we need non-well-founded set theory for this construction because for each situations s, $Rsss$. This means that (s, s) is in $R(s)$ and so, s is not a well-founded set.

We say that $s \trianglelefteq t$ if and only if $SOA(s) \subseteq SOA(t)$ and $R(t) \subseteq R(s)$. The first half of this definition seems obvious. If a situation s contains less information than t, then all of the states of affairs that hold in s also hold in t. The second half might seem less intuitive, but it says something similar. The fewer pairs of situations related to a given situation u, the more implicational propositions hold at u. Thus, if $R(t)$ is a subset of $R(s)$ then at least as many implications hold at t as hold at s.

After defining in such a way the notion of situation we can immediately proceed with the description of situational semantics which in some respects will be similar to that for Dishkant logic. We start from the definition of the frame [3, p. 210].

An $\boldsymbol{R}$-frame is a quadruple $\langle Sit, Logical, R, C\rangle$ such that Sit is a non-empty set, $Logical$ is a non-empty subset of Sit, and R is a three-place arbitrary place relation on Sit, C is a binary relation on Sit, which satisfies the following definitions and postulates:

- $s \trianglelefteq t =_{def} \exists x(x \in Logical \ \& \ Rxst)$;
- $R^1 stu =_{def} Rstu$;
- $R^{n+1}s_1 ... s_{n+2}t$ iff $\exists x(R^n s_1 ... s_{n+1}x \ \& \ Rxs_{n+2}t)$, for $n \geq 1$;
- if $R^n ... s_i, s_{i+1} ... t$, then $R^n ... s_{i+1}, s_i ... t$;
- if $R^n ... s_i ... t$, then $R^{n+1} ... s_i s_i ... t$;
- $Rsss$;
- for each situation s there is a unique $\trianglelefteq$-maximal situation s^* such that Css^*;
- if $Rstu$, then Rsu^*t^*;
- $s^{**} = s$;
- if $s \trianglelefteq t$, then $t^* \trianglelefteq s^*$;
- If $Rbcd$ and $a \trianglelefteq b$, then $Racd$.

Note that worldly situations play no role in this model theory. We can add them. In fact, we can replace the class of logical situations in the specification of a frame with the class of worldly situations and specify that for each worldly situation s, if Cst, then $t \trianglelefteq s$.

Now we proceed with the description of situational model for $\boldsymbol{R}$. A model for $\boldsymbol{R}$ is a pair $\langle \mathfrak{A}, v \rangle$ where $\mathfrak{A}$ is an $\boldsymbol{R}$-frame and v is a value assignment from propositional variables into Sit^2 such that it satisfies *hereditariness*, that is, if $s \in v(p)$ and $s \trianglelefteq t$, then $t \in v(p)$. Each value assignment v determines an interpretation I associated with v, such that:

- $I(p, a) = T$ iff $a \in v(p)$;
- $I(A \wedge B, s) = T$ iff $I(A, s) = T$ and $I(B, s) = T$;
- $I(A \vee B, s) = T$ iff $I(A, s) = T$ or $I(B, s) = T$;
- $I(\neg A, s) = T$ iff $\forall x(Csx \supset I(A, x) = F)$;
- $I(A \rightarrow B, s) = T$ iff $\forall x \forall y(Rsxy \;\&\; I(A, x) = T \supset I(B, y) = T)$.

Comparing given situational semantics with that for non-fregean logic and that for Dishkant logic one can conclude that these three semantics share some common features. So, for example, $\boldsymbol{R}$-frame and $\boldsymbol{LQ}$-frame have, in principle, the same postulates for ternary relation R, the notion of situation for non-fregean logic coincides with the state of affairs in situational $\boldsymbol{R}$-semantics etc. But the difference is also evident.

Proposition 1. *A formula F is valid in $\boldsymbol{R}$ if and only if F is valid in all those $\boldsymbol{R}$-frames $\langle K, O, R, C \rangle$ where K is finite.*

Proof. Let $\Pi = \langle K, O, R, C, I \rangle$ and let $V_F = \{p_1, ..., p_n\}$ be the propositional variables occurring in F. Moreover, let $\mathcal{B}_F$ be the set of all bi-valued assignments $I_F : V_F \rightarrow \{0, 1\}$. We write I_F^a if $\forall p \in V_F(I_F(p) = I(p, a))$ and define a new model $\Pi_f = \langle K_f, O', R', C', I' \rangle$ as follows:

- $K_f = \{I_F \in \mathcal{B}_F : \exists a \in K : I_F = I_F^a\}$
- $I'(p, I_F) = I(p, a)$,where $I_F = I_F^a$
- $R' \subseteq K_f \times K_f \times K_f$ where we take $R' I_F^a I_F^b I_F^c$ as corresponding to $Rabc$.

We can uniquely extend this to all subsets of K_f. It is straightforward to check that for $a \in O$ $I(F, a) = I'(F, I_F)$. Thus we have shown that in evaluating F it suffices to consider Π_f with at most $2^{p(F)}$ where $p(F)$ is the number of different propositional variables occurring in F. □

3. Situational game theoretical semantics for *R*

Finally, bearing in mind game theoretical semantics for non-fregean logic and that for Dishkant logic, we can introduce game theoretical semantics for ***R***.

Assume that two players agree to pay 1€ to the opponent player for each assertion of an atomic statement, which is false in any $a \in Sit$ according to a randomly chosen set of situations. More formally, given a set of all situations K the *risk value* $\langle x \rangle_K$ associated with a propositional variable x is defined as $\langle x \rangle_K = |\{s : I(x, s) = 0\}|$ i.e. the quantity of situations for which x is false.

Let $x_1, x_2, ..., y_1, y_2...$ denote atomic statements, i.e. propositional variables. By $[x_1, ..., x_m || y_1, ..., y_n]$ we denote an *elementary state* in the game where the 1st — the first player — assert each of the y_i in the multiset $\{y_1, ..., y_n\}$ of atomic statements and the 2nd — the second player — assert each atomic statement $x_i \in \{x_1, ..., x_m\}$. The risk associated with a multiset $X = \{x_1, ..., x_m\}$ of atomic formulas is defined as $\langle x_1, ..., x_m \rangle_K = \sum\limits_{i=1}^{m} \langle x_i \rangle_K$. The risk $\langle\rangle_K$ associated with the empty multiset is 0. $\langle V \rangle_K$ respectively denotes the average amount of payoffs that the 1st player expect to have to pay to the 2nd player according to the above arrangements if he/she asserted the atomic formulas in V. The risk associated with an elementary state $[x_1, ..., x_m || y_1, ..., y_n]$ is calculated from the point of view of the 1st player and therefore the condition $\langle x_1, ..., x_m \rangle_K \geq \langle y_1, ..., y_n \rangle_K$ (success condition) expresses that the 1st player do not expect any loss (but possibly some gain) when betting on the truth of atomic statements.

Now we accept the following game rules:

($R_\wedge$) if a player asserts $(A \wedge B)$ in a situation a, then he must assert A in a situation a or B in a situation a at his opponent's choice;

($R_\vee$) if a player asserts $(A \vee B)$ in a situation a, then he must assert A in a situation a or B in a situation a at his own choice;

($R_\rightarrow$) if a player asserts $(A \rightarrow B)$ in a situation a, and another player asserts A in a situation b, then the first player must assert B in a situation c whenever $Rabc$ for any situations b, c. And *vice versa*, i.e. for the roles of the players switched.

A player may also choose not to attack the opponent's assertions of $A \to B$. The rule reflects the idea that the *meaning of implication* entails the principle that an assertion of "If A then B" obliges one to assert also B if the opponent in a game grants (i.e., asserts) A;

($R_\neg$) if a player asserts $\neg A$ in a situation a, then another player asserts A in a situation a^*. And *vice versa*, i.e. for the roles of the players switched.

Henceforth we will use A^a as shorthand for 'A holds at the situation a' and speak of A as a *formula indexed by* a, accordingly. Note, that we have to deal with indexed formulas also in rules $(R_\to)$ and $(R_\neg)$. However, we don't have to change the rule itself, which will turn out to be adequate independently of the kind of evaluation that we aim at in a particular context. We only need to stipulate that in applying $(R_\to)$ the situational index of $A \to B$ (if there is any) is used for defininig the respective indexes for the subformulas A and B.

We use $[A_1^{a_1}, ..., A_m^{a_m} || B_1^{b_1}, ..., B_n^{b_n}]$ to denote an arbitrary (not necessarily elementary) state of the game, where $\{A_1^{a_1}, ..., A_m^{a_m}\}$ is the multiset of formulas that are currently asserted by the second player, and $\{B_1^{b_1}, ..., B_n^{b_n}\}$ is the multiset of formulas that are currently asserted by the first player. (We don't care about the order in which formulas are asserted.)

A move initiated by the 1st player (1st-move) in state $[\Gamma || \Delta]$ consists in his/her picking of some non-atomic formula A^a from the multiset Γ and proceeding as follows:

- if $A^a = (A_1 \to A_2)^a$ then the 1st may either attack by asserting A_1^b in order to force the 2nd to assert A_2^c in accordance with $(R_\to)$, or admit A^a. In the first case the successor state is $[\Gamma', A_2^c || \Delta, A_1^b]$, in the second case it is $[\Gamma' || \Delta]$, where $\Gamma' = \Gamma - \{A^a\}$;
- if $A^a = (\neg A_1)^a$ then the 1st chooses the point a^* thus forcing the 2nd to assert $A_1^{a^*}$. The successor state is $[\Gamma, A_1^{a^*} || \Delta']$, where $\Delta' = \Delta - \{A^a\}$.

A move intiated by the 2nd player (2nd-move) is symmetric, i.e., with the roles of the 1st and the 2nd interchanged. A run of the game consists in a sequence of states, each resulting from a move in the immediately preceding state, and ending in an elementary state $[x_1^{a_1}, ..., x_m^{a_m} || y_1^{b_1}, ..., y_n^{b_n}]$. The 1st player succeed in this run if this final state fulfills the success condition, i.e., if

$$(1) \quad \sum_{j=1}^{n} \langle y_j^{b_j} \rangle_K - \sum_{i=1}^{m} \langle x_i^{a_i} \rangle_K \leq 0.$$

The term at the left hand side of inequality is an *expected loss* of the 1st player at this state. In other words, the 1st succeed if its expected loss is 0 or even negative, i.e., in fact a gain. The other connectives can be reduced to implication and negation.

4. Adequacy of the game

To show that the considered game indeed characterizes logic $\boldsymbol{R}$, we have to analyse all possible runs of the game starting with some arbitrarily complex assertion by the 1st player. A strategy for the 1st player will be a tree-like structure, where a branch represents a possible run resulting from particular choices made by the 1st player, taking into account all possible choices of the 2nd player in (2- or 1-moves) that are compatible with the rules. We will only have to look at strategies for the 2nd player and thus call a strategy winning if the 1st player succeed in all corresponding runs (according to condition (2)).

Taking into account that by Proposition 1 above we can assume that the set *Sit* of situations is finite. The construction of strategies can be viewed as systematic proof search in an analytic tableau calculus with the following rules:

$$\frac{[\Gamma||\Delta, (A_1 \wedge A_2)^a]}{[\Gamma, A_1^a||\Delta]}(\wedge_{1st}^1) \quad \frac{[\Gamma||\Delta, (A_1 \wedge A_2)^a]}{[\Gamma, A_2^a||\Delta]}(\rightarrow_{1st}^2)$$

$$\frac{[\Gamma||\Delta, (A_1 \wedge A_2)^a]}{[\Gamma, A_1^a||\Delta] \mid [\Gamma, A_2^a||\Delta]}(\wedge_{2nd})$$

$$\frac{[\Gamma||\Delta, (A_1 \vee A_2)^a]}{[\Gamma||\Delta, A_1^a]}(\vee_{2nd}^1) \quad \frac{[\Gamma||\Delta, (A_1 \vee A_2)^a]}{[\Gamma||\Delta, A_2^a]}(\vee_{2nd}^2)$$

$$\frac{[\Gamma||\Delta, (A_1 \vee A_2)^a]}{[\Gamma||\Delta, A_2^a] \mid [\Gamma, A_2^a||\Delta]}(\vee_{1st}) \quad \frac{[\Gamma, (A_1 \rightarrow A_2)^a||\Delta]}{[\Gamma, A_2^c||\Delta, A_1^b]}(\rightarrow_{1st}^1)$$

$$\frac{[\Gamma, (A_1 \rightarrow A_2)^a||\Delta]}{[\Gamma||\Delta]}(\rightarrow_{1st}^2) \quad \frac{[\Gamma||\Delta, (A_1 \rightarrow A_2)^a]}{[\Gamma, A_1^b||\Delta, A_2^c] \mid [\Gamma||\Delta]}(\rightarrow_{2nd})$$

$$\frac{[\Gamma||\Delta,(\neg A)^a]}{[\Gamma,A^{a^*}||\Delta]|}(\neg_{2nd})\ \frac{[\Gamma,(\neg A)^a||\Delta]}{[\Gamma||\Delta,A^{a^*}]}(\neg_{1st})$$

In all rules a can denote any index and for any b, c we have $Rabc$. Note that, in accordance with the definition of a strategy for the 2nd player, his/her choices in the moves induce branching, whereas for the 1st player choices a single successor state that is compatible with the game rules is chosen.

THEOREM 1. *A formula F is valid in $\boldsymbol{R}$ if and only if for every set K of situations the 1st player have a winning strategy for the game starting in game state $[||F]$.*

PROOF. Every run of the game is finite. For every final elementary state $[x_1^{a_1},...,x_m^{a_m}||y_1^{b_1},...,y_n^{b_n}]$ the success condition says that we have to compute the risk $\sum\limits_{j=1}^{n}\langle y_j^{b_j}\rangle_K - \sum\limits_{i=1}^{m}\langle x_i^{a_i}\rangle_K$, where $\langle r^a\rangle_K = I(r,a)$, and check whether the resulting value (in the following denoted by $\langle x_1^{a_1},...,x_m^{a_m}||y_1^{b_1},...,y_n^{b_n}\rangle$) is ≤ 0 to determine whether the 1st player 'win' the game. To obtain the minimal final risk of the 1st player (i.e., his/her minimal expected loss) that the 1st can enforce in any given state s by playing according to an optimal strategy, we have to take into account the supremum over all risks associated with the successor states to s that you can enforce by a choice that you may have in a (2nd- or 1st-)move s. On the other hand, for any of the 1st player choices the 1st can enforce the infimum of risks of corresponding successor states. In other words, we prove that we can extend the definition of the 1st expected loss from elementary states to arbitrary states such that the following conditions are satisfied:

(4.1) $\langle\Gamma,(A\to B)^a||\Delta\rangle_K = \inf\{\langle\Gamma||\Delta\rangle_K, \langle\Gamma,B^c||A^b,\Delta\rangle_K : Rabc\}$

(4.2) $\langle\Gamma,(\neg A)^a||\Delta\rangle_K = \sup\{\langle\Gamma||\Delta,A^{a^*}\rangle_K\}$

(4.3) $\langle\Gamma,(A\wedge B)^a||\Delta\rangle_K = \inf\{\langle\Gamma,A^a||\Delta\rangle_K, \langle\Gamma,B^a||\Delta\rangle_K\}$

(4.4) $\langle\Gamma,(A\vee B)^a||\Delta\rangle_K = \sup\{\langle\Gamma,A^a||\Delta\rangle_K, \langle\Gamma,B^a||\Delta\rangle_K\}$

for assertions by the 2nd player and, for assertions by the 1st player:

(4.5) $\langle\Gamma||(A\to B)^a,\Delta\rangle_K = \sup\{\langle\Gamma,A^b||B^c,\Delta\rangle_K, \langle\Gamma||\Delta\rangle_K : Rabc\}$

(4.6) $\langle\Gamma||\Delta,(\neg A)^a\rangle_K = \inf\{\langle\Gamma,A^{a^*}||\Delta\rangle_K\}\rangle$

(4.7) $\langle \Gamma, (A \wedge B)^a || \Delta \rangle_K = \sup\{\langle \Gamma || A^a, \Delta \rangle_K, \langle \Gamma || B^a, \Delta \rangle_K\}$

(4.8) $\langle \Gamma, (A \vee B)^a || \Delta \rangle_K = \inf\{\langle \Gamma || A^a, \Delta \rangle_K, \langle \Gamma || B^a, \Delta \rangle_K\}$.

We have to check that $\langle . || . \rangle_K$ is well-defined; i.e., that conditions above together with the definition of my expected loss (risk) for elementary states indeed can be simultaneously fulfilled and guarantee uniqueness. To this aim consider the following generalisation of the truth function for $\boldsymbol{R}$ to multisets G of indexed formulas:

$$I(\Gamma)_K =_{def} \sum_{\substack{A \in \Gamma \\ a \in dom(I(A))}} I(A, a)$$

Note that

$$I(\{A\})_K = I(A)_K =_{def} \sum_{\substack{A \in \Gamma \\ a \in dom(I(A))}} I(A, a) = 1 \text{ iff } \langle || A \rangle_K \leq 0,$$

that is, A is valid in $\boldsymbol{R}$ iff my risk in the game starting with my assertion of A is non-positive. Moreover, for elementary states we have

$\langle x_1^{a_1}, ..., x_m^{a_m} || y_1^{b_1}, ..., y_n^{b_n} \rangle_K = n - m.$

We generalize the risk function to arbitrary observation states by

$\langle \Gamma || \Delta \rangle_K^* =_{def} |\Delta| - |\Gamma|$

and check that it satisfies conditions (4.1)–(4.8). We only spell out two cases. In order to avoid case distinctions let $I(A^a)_K = I(A, a)$. For condition (4.1) we have

$$\begin{aligned}
\langle \Gamma, (A \to B)^a || \Delta \rangle_K^* = |\Delta| - |\Gamma| - 1 + I(A \to B, a) = \\
= \langle \Gamma || \Delta \rangle_K^* - 1 + I(A \to B, a) = \langle \Gamma || \Delta \rangle_K^* - 1 + (I(A, a) \Rightarrow I(B, a)) = \\
= \langle \Gamma || \Delta \rangle_K^* - \inf\{1, 1 - I(A, a) + I(B, a)\} = \\
= \langle \Gamma || \Delta \rangle_K^* - \inf\{1, 1 + \langle B^c || A^b \rangle_K^*\} = \\
= \langle \Gamma || \Delta \rangle_K^* + \inf\{0, \langle B^c || A^b \rangle_K^*\} = \inf\{\langle \Gamma || \Delta \rangle_K^*, \langle \Gamma, B^c || A^b, \Delta \rangle_K^*\}.
\end{aligned}$$

For condition (4.2) we have

$$\begin{aligned}
\langle \Gamma || \Delta, (\neg A)^a \rangle_K^* = |\Delta| - |\Gamma| - 1 + I(\neg A, a) = \langle \Gamma || \Delta \rangle_K^* - 1 + I(\neg A, a) = \\
= \langle \Gamma || \Delta \rangle_K^* - 1 + \sup(1 - I(A, a^*)) = \sup\{\langle \Gamma || \Delta, A^{c^*} \rangle_K^*\}.
\end{aligned}$$

□

Let us define a *regulation* as assignment of labels '*the 2nd player move next*' and '*the 1st player move next*' to game states that obviously constrain the possible runs of the game. A regulation is consistent if the label '*2nd*(*Ist*) *move next*' is only assigned to states where such a move is possible, i.e., where 1st player (2nd player) have asserted a non-atomic formula. As a corollary to our proof of Theorem, we obtain:

COROLLARY 1. *The total expected loss* $\langle\Gamma||\Delta\rangle^*_K$ *that the 1st player can enforce in a game over* K *starting in state* $[\Gamma||\Delta]$ *only depends on* Γ, Δ *and* K*. In particular, it is the same for every consistent regulation that may be imposed on the game.*

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