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On the Definitional Embeddability of the Combinatory Logic Theory into the First-Order Predicate Calculus

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In this article we prove a theorem on the definitional embeddability of the combinatory logic into the first-order predicate calculus without equality. Since all efficiently computable functions can be represented in the combinatory logic, it immediately follows that they can be represented in the first-order classical predicate logic. So far mathematicians studied the computability theory as some applied theory. From our theorem it follows that the notion of computability is purely logical. This result will be of interest not only for logicians and mathematicians but also for philosophers who study foundations of logic and its relation to mathematics.

Keywords: combinatory logic, definitions, definitional embedding, predicate logic, computable functions

The *signature* Σ is a set of functional and predicate symbols.

The *first order language* $L(\Sigma)$ is the set of formulas of the signature Σ.

The *models* are pairs $M = < D, I >$, where D is a nonempty set of individuals and I is the function of interpretation of functional and predicate symbols.

In standard way we define relations $M, g \models A$ — «*formula A is true in the model M with assignment of values to individual variables g*» and $M \models A$ — «*formula A is true in the model M*».

The *theory in the language* $L(\Sigma)$ is some set of logical and non-logical axioms closed under derivability.

The *first-order predicate calculus in the language* $L(\Sigma)$ is the theory in the language $L(\Sigma)$ with empty set of non-logical axioms.

1. Defining new predicate symbols

We can use definitions in order to extend the language $L(\Sigma)$ of theories with new predicate symbols. The definitions have the following form [2, p. 15]:

$$\forall x_1 \dots x_n (P(x_1, \dots, x_n) \equiv A(x_1, \dots, x_n)).$$

The definition must satisfy the conditions:

1. $P \notin \Sigma$.
2. $A(x_1, \dots, x_n) \in L(\Sigma)$.
3. The variables $x_1, \dots, x_n$ are pairwise distinct.
4. The set of free variables of $A(x_1, \dots, x_n)$ is included into $\{x_1, \dots, x_n\}$.

After defining of the new predicate symbol P, it must be added to the signature Σ. As a result, there is a transition from the language $L(\Sigma)$ to the language $L(\Sigma \cup \{P\})$.

In the language of the first order predicate calculus we can define the universal n-ary predicate U^n by the following definition:

(DU) $\quad \forall x_1 \dots x_n (U^n x_1, \dots, x_n \equiv Px_1 \vee \neg Px_1)$.

The definition allows us to prove $DU \vdash \forall x_1 \dots x_n U^n x_1, \dots, x_n$.

This example is interesting because in the right part of the definition we use an arbitrary predicate symbol of the language of the first order predicate calculus. As another example, we can give a definition of a symmetric relation. Let B be an arbitrary predicate symbol of the language. We accept the following definition:

(DS) $\quad \forall xy(Sxy \equiv \forall uv(Buv \supset Bvu) \supset Bxy)$.

Let us show that $DS \vdash \forall xy(Sxy \supset Syx)$.

1. Sxy	– hyp
2. $\forall uv(Buv \supset Bvu) \supset Bxy$	– from 1 by DS
3. $\forall uv(Buv \supset Bvu)$	– hyp
4. Bxy	– from 2, 3
5. $Bxy \supset Byx$	– from 3
6. Byx	– from 4, 5
7. $\forall uv(Buv \supset Bvu) \supset Byx$	– from 3-6
8. Syx	– from 7 by DS
9. $Sxy \supset Syx$	– from 1-8

It turns out that many well-known theories can be defined within the first order predicate calculus.

DEFINITION 1. The first-order theory T in a language $L(\Sigma)$ with finite set of non-logical axioms Ax is *definitionally embeddable* into predicate calculus, iff there are a signature Σ' and a set of definitions DT of symbols $\Sigma \setminus \Sigma'$ by formulas of $L(\Sigma')$ which met the following condition:

$$\text{If } A \in L(\Sigma), \text{ then } Ax \vdash A \Leftrightarrow DT \vdash A.$$

This definition is some variant of the notion of *definitional embeddability of theories* which was proposed by V.A. Smirnov [3].

2. Combinatory logic as the first-order theory

We can formulate the first order theory of combinatory logic in the language with signature $\Sigma_C = \{\mathbf{K}, \mathbf{S}, *, \geq\}$, where

- $\mathbf{K}$, $\mathbf{S}$ — individual constants.
- $*$ — 2-argument functional symbol.
- $\geq$ — 2-place predicate symbol.

The next formulas are non-logical axioms of the theory:

Ax.1 $\forall xy((\mathbf{K} * x) * y \geq x)$

Ax.2 $\forall xyz(((\mathbf{S} * x) * y) * z \geq (x * z) * (y * z))$

Ax.3 $\forall x(x \geq x)$

Ax.4 $\forall xyz(x \geq y \supset (x * z) \geq (y * z))$

Ax.5 $\forall xyz(x \geq y \supset (z * x) \geq (z * y))$

Ax.6 $\forall xyz(x \geq y \& y \geq z \supset x \geq z)$.

The rules of inference are *modus ponens* and *generalisation*.

It is easy to verify that all the axioms and the rules of inference of combinatory logic as calculus of reductions [1] are provable in this theory.

3. The theorem of embeddability

THEOREM 1. *The theory of combinatory logic is definitionally embeddable into first-order predicate calculus.*

PROOF. We assume that the signature Σ_P of the first-order predicate calculus contains the symbols $\mathbf{K}, \mathbf{S}, *, >$ but doesn't contain the symbol $\geq$.

- $\mathbf{K}$, $\mathbf{S}$ — individual constants.
- $*$ — 2-argument functional symbol.
- $>$ — 2-place predicate symbol.

We write $Ax^{\&>}$ for conjunctions of the axioms Ax.1–6 in which the symbol $\geq$ is renamed into $>$.

We accept the following definition:

$(DC) \qquad \forall xy(x \geq y \equiv (Ax^{\&>} \supset x > y)).$

Let's check $DC \vdash Ax$.1–6.

Ax.1 $\forall xy((\mathbf{K} * x) * y \geq x)$

1. $\vdash Ax^{\&>} \supset \forall xy((\mathbf{K} * x) * y > x)$ – by def. of $Ax^{\&>}$
2. $\vdash \forall xy(Ax^{\&>} \supset (\mathbf{K} * x) * y > x)$ – from 1
3. $DC \vdash \forall xy((\mathbf{K} * x) * y \geq x)$ – from 2 by DC

Ax.2 $\forall xyz(((\mathbf{S} * x) * y) * z \geq (x * z) * (y * z))$

1. $\vdash Ax^{\&>} \supset \forall xyz(((\mathbf{S} * x) * y) * z > (x * z) * (y * z))$ – by def. of $Ax^{\&>}$
2. $\vdash \forall xyz(Ax^{\&>} \supset ((\mathbf{S} * x) * y) * z > (x * z) * (y * z))$ – from 1
3. $DC \vdash \forall xyz(((\mathbf{S} * x) * y) * z \geq (x * z) * (y * z))$ – from 2 by DC

Ax.3 $\forall x(x \geq x)$

1. $\vdash Ax^{\&>} \supset \forall x(x > x)$ – by def. of $Ax^{\&>}$
2. $\vdash \forall x(Ax^{\&>} \supset x > x)$ – from 1
3. $DC \vdash \forall x(x \geq x)$ – from 2 by DC

Ax.4 $\forall xyz(x \geq y \supset (x * z) \geq (y * z))$

1. $\vdash Ax^{\&>} \supset \forall xyz(x > y \supset (x * z) > (y * z))$ – by def. of $Ax^{\&>}$
2. $\vdash \forall xyz(Ax^{\&>} \supset (x > y \supset (x * z) > (y * z)))$ – from 1
3. $\vdash \forall xyz((Ax^{\&>} \supset x > y) \supset (Ax^{\&>} \supset (x * z) > (y * z)))$ – from 2 by self-distr. of $\supset$
4. $DC \vdash \forall xyz(x \geq y \supset (x * z) \geq (y * z))$ – from 3 by DC

Ax.5 $\forall xyz(x \geq y \supset (z * x) \geq (z * y))$

1. $\vdash Ax^{\&>} \supset \forall xyz(x > y \supset (z * x) > (z * y))$ – by def. of $Ax^{\&>}$
2. $\vdash \forall xyz(Ax^{\&>} \supset (x > y \supset (z * x) > (z * y)))$ – from 1
3. $\vdash \forall xyz((Ax^{\&>} \supset x > y) \supset (Ax^{\&>} \supset (z * x) > (z * y)))$ – from 2 by self-distr. of $\supset$
4. $DC \vdash \forall xyz(x \geq y \supset (z * x) \geq (z * y))$ – from 3 by DC

Ax.6 $\forall xyz(x \geq y \& y \geq z \;\supset\; x \geq z)$

1. $\vdash Ax^{\&>} \supset \forall xyz(x > y \& y > z \;\supset\; x > z)$ – by def. of $Ax^{\&>}$
2. $\vdash Ax^{\&>} \supset \forall xyz(x > y \supset (y > z \supset x > z))$ – from 1
3. $\vdash \forall xyz(Ax^{\&>} \supset (x > y \supset (y > z \supset x > z)))$ – from 2
4. $\vdash \forall xyz((Ax^{\&>} \supset x > y) \supset ((Ax^{\&>} \supset y > z) \supset (Ax^{\&>} \supset x > z)))$ – from 3 by self-distr. of $\supset$
5. $\vdash \forall xyz(x \geq y \supset (y \geq z \supset x \geq z))$ – from 4 by DC
6. $DC \vdash \forall xyz(x \geq y \& y \geq z \;\supset\; x \geq z)$ – from 5

Thus we have shown that all the axioms of the theory of combinatory logic are derivable from the DC. It follows that if $A \in L(\Sigma_C)$ and $Ax.1-6 \vdash A$, then $DC \vdash A$.

Now we must show that if $A \in L(\Sigma_C)$ and $DC \vdash A$, then $Ax.1-6 \vdash A$.

Let us assume that $DC \vdash A$ but not $Ax.1-6 \vdash A$. By the completeness theorem of the first-order logic it follows that $DC \models A$ and there exists such a model $M =< D, I >$ and an assignment of values to individual variables g that $M \models Ax.1-6$ and $M, g \models \neg A$.

We can extend the model $M = < D, I >$ to the model $M' = < D, I' >$ in which the formula DC is true. It is sufficient to expand the domain of the function I so that the new function of interpretation I' ascribes value I'$(>) = I(\geq)$ to predicate symbol $>$, and for all other functional and predicate symbols it retains the same value as I.

Since $M \models Ax.1{-}6$, then in the model $M' =< D, I' >$ by the definition of I' we will have $M' \models Ax^{\&>}$ and hence $M' \models x \geq y \equiv (Ax^{\&>} \supset x > y)$. Therefore by our assumption $DC \models A$ it must be $M', g \models A$. However the formula A doesn't contain the symbol $>$, while all other descriptive symbols are interpreted in the same way as in the model M, and by our assumption $M, g \models \neg A$, it must be $M', g \models \neg A$. We have obtained the contradiction. Therefore the assumption that $Ax.1 - 6 \vdash$ A does not hold, is false. □

Corollary 1. *All effectively computable functions can be represented in the first-order predicate logic.*

Proof. This statement is true since all effectively computable functions can be represented in combinatory logic, which, as we have just shown, is definitionally embeddable in the first-order predicate logic. □

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