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## On the ‘classical’ operations in three-valued logics<sup>1</sup>

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The general aim of the present paper is to provide the analysis of the connection between proof-theoretical and functional properties of certain logical matrices. To be more precise, we consider the class of three-valued matrices that induce the classical consequence relation and show that their operations always constitute a subset of one of the maximal classes of functions, which preserve non-trivial equivalence relations. We use a matrix with the single designated value as a sample for in-depth analysis, and generalize the results to suit other cases. Furthermore, on the basis of obtained results we conclude the paper with methodological considerations concerning the nature and interpretation of the truth-values in logical matrices.

*Keywords:* classical propositional logic, three-valued logics, consequence relation, maximal classes of functions, logical matrices

### 1. Introduction

The results presented below belong to the intersection of two prominent fields of modern logic, theory of logical calculi and algebra of logic. The problem that we deal with can be generally described in the form of a question: can we establish the link between a logical consequence relation and the algebraic properties of a matrix which induces it? For the standard two-valued matrix of the classical propositional calculus ( $\mathbf{K}$ ) the answer is clear, as it is a well known fact that Boolean algebra is the algebra of  $\mathbf{K}$ , the set of its basic operations is complete in  $P_2$ , and it contains countable-many closed classes of functions [10, 5]. However, none of the above is the case if we consider the three-valued matrices for  $\mathbf{K}$ . No three-valued Boolean algebras exist, the consequence relation in Post’s three-valued logic is different from the classical one, and, as we will show in the sequel, there are matrices for  $\mathbf{K}$ , which contain continuum-many subclasses. The differences between two-valued and three-valued matrices for  $\mathbf{K}$  make it of interest to investigate the functional properties of the latter. Such an investigation constitutes the subject of the presented research.

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The structure of the paper is as follows. First, we define the necessary concepts, including one of a matrix for an arbitrary propositional language, which induces the classical consequence relation. Then, we present the necessary and sufficient conditions for a three-valued matrix to induce the classical consequence relation. In what follows, we analyze functional properties of the matrices that fulfill this condition, and show the connection of the basic operations of such matrices to the maximal classes of  $P_3$ . The final section of the paper is dedicated to the theoretical analysis of the technical results we have obtained.

## 2. Three-valued matrices which induce the classical consequence relation

We define a *propositional (sentential) language* as an algebra  $\mathcal{S} = \langle S, \xi_1, \xi_2, \dots, \xi_n \rangle$ , where  $S$  is the set of formulae, and  $\xi_1, \xi_2, \dots, \xi_n$  are functions on  $S$ . We will assume that  $a(\xi_i^k) = k \geq 1$  for at least some  $1 \leq i \leq n$ . Given a set  $Var(\mathcal{S}) = \{p_1, p_2, \dots, p_i, \dots\}$  of the propositional variables of  $\mathcal{S}$ , we define the contents of  $S$  inductively:

- If  $\alpha \in Var(\mathcal{S})$ , then  $\alpha \in S$ ;
- if  $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subseteq S$  and  $\xi_i^k \in \mathcal{S}$  ( $a(\xi_i^k) = k$ ), then  $\xi_i^k(\alpha_1, \alpha_2, \dots, \alpha_k) \in S$ ;
- there are no other elements in  $S$ .

A *logical matrix*  $\mathcal{M} = \langle \mathcal{A}, D \rangle$  is a structure, where  $\mathcal{A} = \langle A, F \rangle$  is an algebra, and  $D$  is a non-empty proper subset of  $A$ . The elements of  $D$  will be referred to as *designated values*. If  $\langle \mathcal{S}, \xi_1, \dots, \xi_n \rangle$  and  $\langle \mathcal{A}, F \rangle$  are of the same type, then  $\mathcal{M}$  is a *matrix for*  $\mathcal{S}$ , and a homomorphism  $v$  from  $S$  into  $A$  will be called a *valuation* of  $\mathcal{S}$ -formula in  $\mathcal{M}$ .

By *consequence relation induced by*  $\mathcal{M}$  we will denote the set  $\vDash_{\mathcal{M}} = \{\langle X, \alpha \rangle \mid X \vDash_{\mathcal{M}} \alpha\}$ , where  $X \vDash_{\mathcal{M}} \alpha$  ( $\{X \cup \alpha\} \subseteq S$ ) iff for every valuation  $v$  in  $\mathcal{M}$  it is true that  $v(\alpha) \in D$  whenever  $v(X) \subseteq D$ .

Let  $\mathcal{S}$  be such a language that there is a matrix  $\mathcal{K} = \langle \{0, 1\}, F_{\mathcal{K}}, \{1\} \rangle$  for  $\mathcal{S}$ , where  $[F_{\mathcal{K}}] = P_2$ , where  $[F_{\mathcal{K}}]$  is the closure of  $F_{\mathcal{K}}$  under Mal'tsev operations [8]. We will say that the consequence relation induced by a matrix  $\mathcal{M}$  for  $\mathcal{S}$  is *classical* iff  $\vDash_{\mathcal{M}} = \vDash_{\mathcal{K}}$ .

Now we need to introduce the concept of *matrix homomorphism* [15, 9]. Let  $\mathcal{M} = \langle \mathcal{A}, D \rangle$  and  $\mathcal{M}' = \langle \mathcal{A}', D' \rangle$  be matrices of the same type. A homomorphism  $h$  from  $\mathcal{A}$  into  $\mathcal{A}'$  is said to be a homomorphism from  $\mathcal{M}$

into  $\mathcal{M}'$  iff  $h(D) \subseteq D'$ . If it also holds that  $h^{-1}(D') = D$ ,  $h$  is said to be a matrix (or strong) homomorphism.

The following theorem can be proved [1]:

**THEOREM 1.** *Let  $\mathcal{M}_3 = \langle \{0, 1, 2\}, F, D \rangle$  be a matrix of the same type as  $\mathcal{K}$ . Then  $\models_{\mathcal{M}_3} = \models_{\mathcal{K}}$  iff there is a matrix homomorphism from  $\mathcal{M}_3$  into  $\mathcal{K}$ .*

In other words, the consequence relation induced by a three-valued matrix  $\mathcal{M}_3$  is classical iff for every  $n$ -ary function of its algebra it is true that

$$h(f(a_1, a_2, \dots, a_n)) = f_{\mathcal{K}}(h(a_1), h(a_2), \dots, h(a_n)),$$

where  $h(a_i) = 1$ , if  $a_i \in D$ , and  $h(a_i) = 0$  otherwise, for every  $a_i \in \{0, 1, 2\}$  ( $1 \leq i \leq n$ ).

In the usual manner, we can make a transition from the matrix homomorphism to the matrix congruence:

$$\langle a_1, a'_1 \rangle \in \kappa_h \Leftrightarrow h(a_1) = h(a'_1).$$

As we have limited ourselves to the three-valued case, the matrices we consider can only differ by the elements of  $[F]$  and  $D$ . Obviously, the choice of  $D$  impacts the structure of  $\kappa_h$ . For example,

- if  $D = \{2\}$ , then  $\langle 0, 1 \rangle \in \kappa_h$ ;
- if  $D = \{1, 2\}$ , then  $\langle 1, 2 \rangle \in \kappa_h$ .

As soon as we determine, what is the class of designated values of  $\mathcal{M}_3$  which determines the structure of matrix congruence  $\kappa_h$  on  $\mathcal{M}_3$ , the contents of  $[F]$  become the only variable.

This allows us to introduce a concept of the *classical functions* on  $\{0, 1, 2\}$ . It will be said that an  $n$ -ary function  $f$  on  $\{0, 1, 2\}$  is *classical in respect to  $D$*  iff it satisfies the following condition:

$$\begin{aligned} & \{ \langle a_1, a'_1 \rangle, \langle a_2, a'_2 \rangle, \dots, \langle a_n, a'_n \rangle \} \subseteq \kappa_h \Rightarrow \\ & \langle f(a_1, a_2, \dots, a_n), f(a'_1, a'_2, \dots, a'_n) \rangle \in \kappa_h, \end{aligned}$$

where  $\kappa_h$  depends on the contents of  $D$ . Obviously, if  $\models_{\mathcal{M}_3} = \models_{\mathcal{K}}$ , then all functions from  $[F]$  are classical.

In the following section we shall investigate the properties of the class of functions which are classical in respect to  $D = \{2\}$ , and the matrix which contains all of such operations.

### 3. The maximal three-valued matrix with operations which are classical in respect to $D = \{2\}$

S.V. Jablonskij has described all 18 classes of functions maximal in  $P_3$ , including the three classes of type  $U$ , the maximal classes of functions, which preserve non-trivial equivalence relations [4]. For us, of special interest is the class  $U_2$ , which is defined as follows:

$f(x_1, x_2, \dots, x_n) \in U_2$ , iff for all  $1 \leq i_1 < i_2 < \dots < i_s \leq n$  on all sets of values  $\{b_1, b_2, \dots, b_n\}$ , where

$$b_m = \begin{cases} 2, & \text{if } m = i_l \ (l = 1, 2, \dots, s), \\ \neq 2 & \text{otherwise,} \end{cases}$$

function  $f(x_1, x_2, \dots, x_n)$  either returns values from  $\{0, 1\}$  exclusively, or is equivalent to 2.

One can observe that

$$F_2 = \{f(x_1, x_2, \dots, x_n) | \{\langle a_1, a'_1 \rangle, \langle a_2, a'_2 \rangle, \dots, \langle a_n, a'_n \rangle\} \subseteq \kappa_2 \Rightarrow \langle f(a_1, a_2, \dots, a_n), f(a'_1, a'_2, \dots, a'_n) \rangle \in \kappa_2\} \text{ coincides with } U_2.$$

In other words,  $U_2$  is exactly the class of functions which are classical in respect to  $D = \{2\}$ .

Now we will build a three-valued matrix containing all functions which are classical in respect to  $D = \{2\}$ . First, let us consider some operations and show that each of them satisfies the definition of a function from  $U_2$ .

$f_\wedge$	2	1	0
2	2	1	0
1	1	1	0
0	0	0	0

Suppose  $s = 0$ . We have four value sets:  $\langle 0, 0 \rangle$ ,  $\langle 0, 1 \rangle$ ,  $\langle 1, 0 \rangle$ ,  $\langle 1, 1 \rangle$ , and  $f_\wedge(0, 0) = f_\wedge(0, 1) = f_\wedge(1, 0) = 0$ ,  $f_\wedge(1, 1) = 1$ .

Suppose  $s = 1$ . For  $i_1 = 1$  we have  $\langle 2, 0 \rangle$ ,  $\langle 2, 1 \rangle$ . For  $i_1 = 2$  we have  $\langle 0, 2 \rangle$ ,  $\langle 1, 2 \rangle$ . And  $f_\wedge(2, 0) = f_\wedge(0, 2) = 0$ ,  $f_\wedge(2, 1) = f_\wedge(1, 2) = 1$ .

For  $s = 2$  we have one value set:  $\langle 2, 2 \rangle$ , and  $f_\wedge(2, 2) = 2$ .

$f_\vee$	2	1	0
2	2	2	2
1	2	1	1
0	2	1	0

Suppose  $s = 0$ . We have four value sets:  $\langle 0, 0 \rangle$ ,  $\langle 0, 1 \rangle$ ,  $\langle 1, 0 \rangle$ ,  $\langle 1, 1 \rangle$ , and  $f_{\vee}(0, 0) = 0$ ,  $f_{\vee}(0, 1) = f_{\vee}(1, 0) = f_{\vee}(1, 1) = 1$ .

Suppose  $s = 1$ . For  $i_1 = 1$  we have  $\langle 2, 0 \rangle$ ,  $\langle 2, 1 \rangle$ . For  $i_1 = 2$  we have  $\langle 0, 2 \rangle$ ,  $\langle 1, 2 \rangle$ . And  $f_{\vee}(2, 0) = f_{\vee}(2, 1) = f_{\vee}(0, 2) = f_{\vee}(1, 2) = 2$ .

For  $s = 2$  we have one value set:  $\langle 2, 2 \rangle$ , and  $f_{\vee}(2, 2) = 2$ .

$f_{\supset}$	2	1	0
2	2	0	1
1	2	2	2
0	2	2	2

Suppose  $s = 0$ . We have four value sets:  $\langle 0, 0 \rangle$ ,  $\langle 0, 1 \rangle$ ,  $\langle 1, 0 \rangle$ ,  $\langle 1, 1 \rangle$ , and  $f_{\supset}(0, 0) = f_{\supset}(0, 1) = f_{\supset}(1, 0) = f_{\supset}(1, 1) = 2$ .

Suppose  $s = 1$ . For  $i_1 = 1$  we have  $\langle 2, 0 \rangle$ ,  $\langle 2, 1 \rangle$ . For  $i_1 = 2$  we have  $\langle 0, 2 \rangle$ ,  $\langle 1, 2 \rangle$ . And  $f_{\supset}(2, 0) = 1$ ,  $f_{\supset}(2, 1) = 0$ ,  $f_{\supset}(0, 2) = f_{\supset}(1, 2) = 2$ . For  $s = 2$  we have one value set:  $\langle 2, 2 \rangle$ , and  $f_{\supset}(2, 2) = 2$ .

$x_1$	$f_{\neg}(x_1)$
2	0
1	2
0	2

For  $s = 0$  we have  $f_{\neg}(0) = f_{\neg}(1) = 2$ . For  $s = 1$  we have  $f_{\neg}(2) = 0$ .

Now, consider the matrices  $\mathcal{M}_{max}^1 = \langle \{0, 1, 2\}, \mathcal{F}_{max}^1, \{2\} \rangle$ , where  $\mathcal{F}_{max}^1 = \{f_{\wedge}, f_{\vee}, f_{\supset}, f_{\neg}\}$ , and  $\mathcal{M}_{\mathcal{K}} = \langle \{0, 1\}, \mathcal{F}_{\mathcal{K}}, \{1\} \rangle$ , where  $\mathcal{F}_{\mathcal{K}} = \{g_{\wedge}, g_{\vee}, g_{\supset}, g_{\neg}\}$ .

$g_{\wedge}$	1	0	$g_{\vee}$	1	0	$g_{\supset}$	1	0	$x$	$g_{\neg}(x)$
1	1	0	1	1	1	1	1	0	1	0
0	0	0	0	1	0	0	1	1	0	1

A propositional language  $\mathcal{L} = \langle L, \wedge, \vee, \supset, \neg \rangle$  is said to be standard. Both  $\mathcal{M}_{max}^1$  and  $\mathcal{M}_{\mathcal{K}}$  are matrices for  $\mathcal{L}$ . Moreover,  $[\mathcal{F}_{\mathcal{K}}] = P_2$ . It is easy to check that  $\models_{\mathcal{M}_{max}^1} = \models_{\mathcal{M}_{\mathcal{K}}}$ . Therefore, the consequence relation induced by  $\mathcal{M}_{max}^1$  is classical.

LEMMA 1.  $[\mathcal{F}_{max}^1]$  is maximal in  $P_3$ .

In other words,

$$\forall f(x_1, x_2, \dots, x_n)(f(x_1, x_2, \dots, x_n) \notin [\mathcal{F}_{max}^1] \Rightarrow [\mathcal{F}_{max}^1 \cup \{f(x_1, x_2, \dots, x_n)\}] = P_3),$$

where  $P_3$  is the class of all functions on  $\{0, 1, 2\}$ , and  $f(x_1, x_2, \dots, x_n) \in P_3$ .

PROOF. It is sufficient to show that every function, which is classical in respect to  $\{2\}$ , is equivalent to a superposition of the functions of  $\mathcal{F}_{max}^1$ .

The following functions belong to  $[\mathcal{F}_{max}^1]$  (to simplify the notation, we write « $\xi$ » instead of « $f_\xi$ », where  $\xi \in \{\wedge, \vee, \supset, \neg\}$ ).

- $\blacktriangle(x) = (x \supset x) \supset x$ ;
- $I'_2(x) = \neg\neg x \wedge \blacktriangle\neg x$ ;
- $I'_1(x) = \blacktriangle((\neg x \wedge \blacktriangle x) \vee (\neg\neg x \wedge \blacktriangle\neg x))$ ;
- $I'_0(x) = \neg x \wedge \blacktriangle x$ .

$x$	$\blacktriangle$	$I'_2$	$I'_1$	$I'_0$
2	2	1	0	0
1	0	0	1	0
0	1	0	0	1

Consider the function

$$\bigwedge_{i=1}^n I_{a_i}(x_i) = I_{a_1}(x_1) \wedge I_{a_2}(x_2) \wedge \dots \wedge I_{a_n}(x_n),$$

where  $I_{a_i}(x_i) = \neg\neg x_i$ , if  $a_i = 2$ , and  $I_{a_i}(x_i) = \neg x_i$ , if  $a_i \in \{0, 1\}$ . The function  $\bigwedge_{i=1}^n I_{a_i}(x_i)$  produces the value 2, if  $x_i = a_i = 2$ , or  $x_i \in \{0, 1\}$  and  $a_i \in \{0, 1\}$ , and it produces the value 0 otherwise.

Moreover, the function

$$\bigvee_{i=1}^n I'_{a_i}(x_i) = I'_{a_1}(x_1) \vee I'_{a_2}(x_2) \vee \dots \vee I'_{a_n}(x_n)$$

produces the value 1, if  $x_i = a_i$  for every  $i$ , and the value 0 otherwise.

Let  $f(x_1, x_2, \dots, x_n)$  be a function which is classical in respect to  $\{2\}$ . Assume  $f(x_1, x_2, \dots, x_n) = 2$ , only when  $x_i = b_{j_i}$  ( $1 \leq j \leq k$ ) for the sets of values  $(b_{1_1}, b_{1_2}, \dots, b_{1_n})$ ,  $(b_{2_1}, b_{2_2}, \dots, b_{2_n})$ ,  $\dots$ ,  $(b_{k_1}, b_{k_2}, \dots, b_{k_n})$ . Since  $f(b_{j_1}, b_{j_2}, \dots, b_{j_n}) = 2$  iff  $f(b_{j_1^*}, b_{j_2^*}, \dots, b_{j_n^*}) = 2$ , where  $b_{j_i^*} = 0$ , if  $b_{j_i} = 1$ , and  $b_{j_i^*} = b_{j_i}$  otherwise,

$$\bigvee_{j=1}^k \left( \bigwedge_{i=1}^n I_{b_{j_i}}(x_i) \right) = \left( \bigwedge_{i=1}^n I_{b_{1_i}}(x_i) \right) \vee \left( \bigwedge_{i=1}^n I_{b_{2_i}}(x_i) \right) \vee \dots \vee \left( \bigwedge_{i=1}^n I_{b_{k_i}}(x_i) \right)$$

is a function, which produces the value 2, if  $f(x_1, x_2, \dots, x_n) = 2$ , and the value 0 otherwise.

Now assume  $f(x_1, x_2, \dots, x_n) = 1$ , only when  $x_i = c_{j'_i}$  ( $1 \leq j' \leq m$ ) for the sets of values  $(c_{1_1}, c_{1_2}, \dots, c_{1_n})$ ,  $(c_{2_1}, c_{2_2}, \dots, c_{2_n})$ ,  $\dots$ ,  $(c_{m_1}, c_{m_2}, \dots, c_{m_n})$ . Then there is a function

$$\bigvee_{j'=1}^m \left( \bigwedge_{i=1}^n I'_{c_{j'_i}}(x_i) \right) = \left( \bigvee_{i=1}^n I'_{c_{1_i}}(x_i) \right) \vee \left( \bigvee_{i=1}^n I'_{c_{2_i}}(x_i) \right) \vee \dots \vee \left( \bigvee_{i=1}^n I'_{c_{m_i}}(x_i) \right),$$

where  $\bigvee_{j'=1}^m \left( \bigwedge_{i=1}^n I_{c_{j'_i}}(x_i) \right)$  produces the value 1, if  $f(x_1, x_2, \dots, x_n) = 1$ , and the value 0 otherwise.

If  $f(x_1, x_2, \dots, x_n) = 1$ , then  $\bigvee_{j=1}^k \left( \bigwedge_{i=1}^n I_{b_{j_i}}(x_i) \right) = 0$ , hence the following holds:

$$\bigvee_{j=1}^k \left( \bigwedge_{i=1}^n I_{b_{j_i}}(x_i) \right) \vee \bigvee_{j'=1}^m \left( \bigwedge_{i=1}^n I'_{c_{j'_i}}(x_i) \right) = f(x_1, x_2, \dots, x_n).$$

Therefore, every function which is classical in respect to  $\{2\}$  is equivalent to a superposition of the functions of  $\mathcal{F}_{max}^1$ . As the class of functions, which are classical in respect to  $\{2\}$ , coincides with  $U_2$ , and  $U_2$  is maximal in  $P_3$ ,  $\mathcal{F}_{max}^1$  is maximal in  $P_3$ .  $\square$

#### 4. Generalizations and analysis

Our definition of the class  $D$  of designated values implies the following options:  $\{0, 1\}$ ,  $\{0, 2\}$ ,  $\{1, 2\}$ ,  $\{0\}$ ,  $\{1\}$ ,  $\{2\}$ . Similarly to the case when  $D = \{2\}$ , the classes of functions which are classical in respect to other sets of designated values ( $[F_D]$ ) coincide with one of the three classes of functions, which preserve non-trivial equivalence relations —  $U_2$ ,  $U_1$ ,  $U_0$ . The classes  $U_2$ ,  $U_1$ , and  $U_0$  are pairwise dual and, therefore, isomorphic [4]. So all results obtained for  $D = \{2\}$  can be easily generalized for other sets of designated values. The relations between the classes of classical functions and the classes of the type  $U$  are as follows:

- $[F_{\{0,1\}}] = [F_{\{2\}}] = U_2$ .
- $[F_{\{0,2\}}] = [F_{\{1\}}] = U_1$ .
- $[F_{\{1,2\}}] = [F_{\{0\}}] = U_0$ .

As shown above, in all three cases  $F_D = U_D$  and  $F_{A \setminus D} = U_D$ . If we adopt the usual view of truth-values as degrees of truth, this can

seem counterintuitive. However, it is perfectly in line with G. Malinowski's observation that we can pick  $D = \{0\}$  in a two-valued matrix [7]. This way we are able obtain two classical two-valued logics — «truth-based» with  $D = \{1\}$ , and «falsity-based» with  $D = \{0\}$ .

Let us also note that even in the case of  $D = \{2\}$  there are classical operations that do not satisfy the «normality» condition (see [3, 13] for discussion of  $C$ -normality and [14] for analysis concerning implication in particular). For example,  $f_{\supset} \in \mathcal{F}_{max}^1$  is a classical implication in terms of the current paper. But  $f_{\supset}(2, 0) = 1$ , and that contradicts the idea that a «normal» implication must preserve the classical truth-values.

Another point worth investigating is the power of the sets of all subclasses of classical functions. The set of functions of two-valued classical logic is complete in  $P_2$ . And  $P_2$  has countable-many subclasses. However, each of  $U_0$ ,  $U_1$ , and  $U_2$  has continuum-many subclasses [6]. Therefore, every maximal set of classical functions has continuum-many subclasses. Moreover, the set of functions of Heyting's three-valued logic  $\mathbf{G}_3$  has continuum-many subclasses as well [12], and it is a proper subset of  $U_0$ . This shows that the fact we pointed out holds even for non-maximal sets of classical operations. Although the sets of classical functions with countable-many subclasses exist as well. Consider the following operations:

$\cap$	2	1	0	$\cup$	2	1	0	$\supset$	2	1	0	$x$	$\neg x$
2	2	0	0	2	2	2	2	2	2	0	0	2	0
1	0	0	0	1	2	0	0	1	2	2	2	1	2
0	0	0	0	0	2	0	0	0	2	2	2	0	2

The matrix  $\langle \{0, 1, 2\}, \cap, \cup, \supset, \neg, \{2\} \rangle$  is a submatrix of Bochvar's three-valued logic  $\mathbf{B}_3$  [2]. As Bochvar pointed out, the fragment of  $\mathbf{B}_3$  determined by this matrix is isomorphic to  $\mathbf{K}$ . Indeed, the class  $[\{\supset, \neg\}]$  ( $\cap$  and  $\cup$  are not independent from  $\{\supset, \neg\}$ ) is a closed class of operations which are classical in respect to  $D = \{2\}$ . While the set of functions of  $\mathbf{B}_3$  itself contains continuum-many subclasses [11], for every function  $f(x_1, x_2, \dots, x_n) \in [\{\supset, \neg\}]$  it is true that  $f(a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = f(a_1, a_2, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n)$  for every  $a_1, a_2, \dots, a_n$  ( $a_j \in \{0, 1, 2\}$ ). Hence,  $[\{\supset, \neg\}]$  is isomorphic to  $P_2$ , and, therefore, contains countable-many closed subclasses.

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