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A Weakly-Intuitionistic Logic $I1$

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In 1995, Sette and Carnielli presented a calculus, $I1$, which is intended to be *dual* to the paraconsistent calculus $P1$. The duality between $I1$ and $P1$ is reflected in the fact that both calculi are maximal with respect to classical propositional logic and they *behave* in a special, non-classical way, but only at the level of variables. Although some references are given in the text, the authors do not explicitly define what they mean by 'duality' between the calculi. For instance, no definition of the translation function from the language of $I1$ into the language of $P1$ (or from $P1$ to $I1$) was provided (see [4], pp. 88–90) nor was it shown that the calculi were functionally equivalent (see [13], pp. 260–261).

The purpose of this paper is to present a new axiomatization of $I1$ and briefly discuss some results concerning the issue of duality between the calculi.

Keywords: weakly-intuitionistic logic, paracomplete logic, $I1$, paraconsistent logic, Sette's calculus, $P1$

1. Introduction

Suppose that **L** is a logic defined in a propositional language with at least the connectives: $\sim$, $\wedge$ and $\vee$. We say that (1) a logic **L** is *weakly-intuitionistic* if the law of excluded middle $\alpha \vee \sim \alpha$ is not valid in **L**; (2) a logic **L** is *weakly-paraconsistent* if the law of non-contradiction $\sim (\alpha \wedge \sim \alpha)$ is not valid in **L** (*cf.* [11], p. 182).

Though this definition is intuitive enough, it may give rise to some doubts. Observe, for example, that intuitionistic logic is weakly-intuitionistic (but not *vice versa*) and there are some paraconsistent logics which are weakly-paraconsistent — it suffices to recall that the law of non-contradiction is not valid in da Costa's calculi C_n (see [2]). On the other hand, some paraconsistent logics such as $CLuNs$ or Jaśkowski's discursive logic are not weakly-paraconsistent at all (see [1], [6] and [7]). So, as we can see, there is a kind of asymmetry here probably caused by the lack of uniform criteria for paraconsistency (*cf.* [8]) — not to mention that one and only one paraconsistent logic does not exist, if any (see [12]).

Another point is that the calculi $I1$ and $P1$ are defined in a propositional language with the connectives of negation and implication taken as primitives. In fact, the connectives of conjunction and disjunction are nothing but abbreviations (see [9, p. 178] and [11, p. 199], for details) which do not appear explicitly in formulas. This leads to an alternative definition of the weakly-intuitionistic (and weakly-paraconsistent) logic, *viz.*

Definition 1. A logic **L** is *weakly-intuitionistic* if the law of Clavius, $(\sim p_1 \to p_1) \to p_1$, is not valid in **L**, for any $p_1 \in var$.

Definition 2. A logic **L** is *weakly-paraconsistent* if the law of Duns Scotus, $p_1 \to (\sim p_1 \to p_2)$, is not valid in **L**, for any $p_1, p_2 \in var$.

where $var = \{p_1, p_2, p_3, ...\}$ and $i \in N$.

Now let us consider the following axiom schemata:

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(PL) $((\alpha \to \beta) \to \alpha) \to \alpha$
(DS) $\alpha \to (\sim \alpha \to \beta)$
(CM) $(\sim \alpha \to \alpha) \to \alpha$

and Detachment, (MP) $\alpha, \alpha \to \beta / \beta$, as the sole rule of inference.

Notice that (MP) plus (A1), (A2), (PL), (DS) and (CM) define classical implication and classical negation (*cf.* [5, p. 437]). This will be a starting point for our analysis, in which a new axiomatization of $I1$ (and $P1$) is proposed.

2. Weakly-Intuitionistic Calculus $I1$

In this section, we present a new axiomatization of the calculus $I1$. The axiom schemata will be chosen to show that $I1$ behaves in *a weakly-intuitionistic* way only at the level of variables, i.e. the so-called *consequentia mirabilis*, $(\sim \alpha \to \alpha) \to \alpha$, is an $I1$-tautology provided that α is not a propositional variable. As will be seen, a new set of axioms for $I1$ is easily obtained from the set given in Section 1 by imposing an additional condition on the axiom (CM).

Let var be a non-empty denumerable set of all propositional variables. The set of all formulas, For, is inductively defined as follows:

(1) $p_i \in For$, where $p_i \in var$ and $i \in N$
(2) if $\alpha \in For$, then $\sim \alpha \in For$
(3) if α and $\beta \in For$, then $\alpha \to \beta \in For$.

The calculus $I1$ is axiomatized by means of the following axiom schemata:

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(A3) $(\sim\sim \alpha \to \sim \beta) \to ((\sim\sim \alpha \to \beta) \to \sim \alpha)$
(A4) $\sim\sim (\alpha \to \beta) \to (\alpha \to \beta)$.

and (MP) α, $\alpha \to \beta$ / β [11, p. 182–183].

Definition 3. A formal proof (deduction) within $I1$ of α from the set formulas of Γ is a finite sequence of formulas, $\beta_1, \beta_2, ...\beta_n$, where $\beta_n = \alpha$ and each of elements in that sequence is either an axiom of $I1$, or belongs to Γ, or follows from the preceding formulas in the sequence by (MP).

Definition 4. A formula α is a syntactic consequence within $I1$ of a set formulas of Γ ($\Gamma \vdash_{I1} \alpha$, in symbols) *iff* there is a formal proof of α from the set Γ within $I1$.

Definition 5. A formula α is a thesis of $I1$ *iff* $\emptyset \vdash_{I1} \alpha$.

Theorem 1. *$\Gamma \vdash_{I1} \alpha \to \beta$ iff $\Gamma \cup \{\alpha\} \vdash_{I1} \beta$, where α, $\beta \in For$, $\Gamma \subset For$.*

Proof. By induction. Apply ($A1$), ($A2$), $\alpha \to \alpha$ and use (MP) as the sole rule of inference. $\square$

Fact 1. The formulas

(TR) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$
(SIM) $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$
(PER) $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
(PL) $((\alpha \to \beta) \to \alpha) \to \alpha$
(DS) $\alpha \to (\sim \alpha \to \beta)$
(NN1) $\alpha \to \sim\sim \alpha$
(NN2v) $\sim\sim \alpha \to \alpha$, if $\alpha \notin var$
(CMv) $(\sim \alpha \to \alpha) \to \alpha$, if $\alpha \notin var$

are provable in $I1$.

Proof. (TR), (SIM), (PER) by Theorem 1 (DT for short) and (MP).

(PL). See [11, p. 188–189].

(DS). *Ibid.*, p. 189–190.

(NN2v). *Ibid.*, p. 183–184.

(NN1).

(a)	α	by DT
(b)	$\alpha \to (\sim\sim\sim \alpha \to \alpha)$	$\{(A1)\}$
(c)	$\sim\sim \alpha \to \alpha$	$\{(MP), (a), (b)\}$
(d)	$\sim\sim\sim \alpha \to \sim \alpha$	$\{(NN2v) : \alpha \notin var\}$
(e)	$(\sim\sim\sim \alpha \to \sim \alpha) \to ((\sim\sim\sim \alpha \to \alpha) \to \sim\sim \alpha)$	$\{(A3)\}$
(f)	$(\sim\sim\sim \alpha \to \alpha) \to \alpha$	$\{(MP), (d), (e)\}$
(g)	α	$\{(MP), (c), (f)\}$.

(CMv). By cases. Let $\alpha \notin var$ then
Case 1: α is of the form $\sim \phi$.

(a)	$\sim\sim \phi \to \sim \phi$	by DT
(b)	$\sim \phi \to (\sim\sim \phi \to \phi)$	$\{(DS)\}$
(c)	$\sim\sim \phi \to (\sim\sim \phi \to \phi)$	$\{(TR), (MP), (a), (b)\}$
(d)	$\sim\sim \phi \to \phi$	$\{MP), (c), (SIM)\}$
(e)	$(\sim\sim \phi \to \sim \phi) \to ((\sim\sim \phi \to \phi) \to \sim \phi)$	$\{(A3)\}$
(f)	$(\sim\sim \phi \to \phi) \to \sim \phi$	$\{(MP), (a), (e)\}$
(g)	$\sim \phi$	$\{(MP), (d), (f)\}$.

Case 2: α is of the form $\phi \to \psi$.

(a)	$\sim (\phi \to \psi) \to (\phi \to \psi)$	by DT
(b)	$\sim\sim\sim (\phi \to \psi) \to \sim (\phi \to \psi)$	$\{(NN2v) : \alpha \notin var\}$
(c)	$\sim\sim\sim (\phi \to \psi) \to (\phi \to \psi)$	$\{(TR), (MP), (b), (a)\}$
(d)	$\sim\sim (\phi \to \psi)$	$\{(A3), (MP), (b), (c)\}$
(e)	$\sim\sim (\phi \to \psi) \to (\phi \to \psi)$	$\{(NN2v) : \alpha \notin var\}$
(f)	$\phi \to \psi$	$\{(MP), (d), (e)\}$.

□

Sette and Carnielli proved that $I1$ was complete with respect to the matrix

$$M_{I1} = \langle \{0, 1, 2\}, \{1\}, \sim, \to \rangle,$$

where $\{0, 1, 2\}$ is the set of logical values, $\{1\}$ contains the designated value and the connectives of implication and negation are defined by the truth-tables:

$\to$	1	2	0
1	1	0	0
2	1	1	1
0	1	1	1

$\sim$	
1	0
2	0
0	1

An $I1$-valuation is any function $v : For \longrightarrow \{1, 2, 0\}$ compatible with the above truth-tables. An $I1$-tautology is a formula which under any valuation v takes on the designated value $\{1\}$.

Observe that neither the formula $(\sim \alpha \rightarrow \alpha) \rightarrow \alpha$ nor $\sim\sim \alpha \rightarrow \alpha$ is an $I1$-tautology.

Let I^* be a calculus axiomatized by

(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$

(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

(PL) $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow \alpha$

(DS) $\alpha \rightarrow (\sim \alpha \rightarrow \beta)$

(CMv) $(\sim \alpha \rightarrow \alpha) \rightarrow \alpha$, if $\alpha \notin var$

and the rule (MP), then

Fact 2. $I^* = I1$.

Proof.

($\subset$) (A1), (A2), (PL), (DS) and (CMv) are theorems of $I1$ (*cf.* Fact 1) and (MP) is the sole rule of inference in $I1$. Then, by soundness, all the formulas are $I1$-tautologies and (MP) preserves validity.

($\supset$) What is desired is to demonstrate that (A3) and (A4) are provable in I^*. To prove this, we first need to show that some additional formulas are provable in I^*, *viz.* (TR), (SIM), (PER), (R) $((\alpha \rightarrow \beta) \rightarrow \alpha) \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha)$, (CMn) $(\sim \alpha \rightarrow \sim\sim \alpha) \rightarrow \sim\sim \alpha$, (NN1) $\alpha \rightarrow \sim\sim \alpha$ and (CON) $(\sim (\alpha \rightarrow \beta) \rightarrow \sim\sim\sim (\alpha \rightarrow \beta)) \rightarrow (\sim\sim (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta))$.

(TR), (SIM), (PER) by DT and (MP).

(R) by DT, (PL) and (MP).

(CMn).

(a)	$\sim \alpha \rightarrow \sim\sim \alpha$	by DT
(b)	$(\sim\sim \alpha \rightarrow \sim \alpha) \rightarrow \sim \alpha$	$\{(CMv) : \sim \alpha/\alpha\}$
(c)	$(\sim \alpha \rightarrow \sim\sim \alpha) \rightarrow ((\sim\sim \alpha \rightarrow \sim \alpha) \rightarrow \sim\sim \alpha)$	$\{(MP), (TR), (b)\}$
(d)	$(\sim\sim \alpha \rightarrow \sim \alpha) \rightarrow \sim\sim \alpha$	$\{(MP), (a), (c)\}$
(e)	$(\sim \alpha \rightarrow \sim\sim \alpha) \rightarrow \sim\sim \alpha$	$\{(MP), (R), (d)\}$
(f)	$\sim\sim \alpha$	$\{(MP), (a), (e)\}$.

(NN1).

(a) α by DT
(b) $\alpha \to (\sim \alpha \to \sim\sim \alpha)$ $\{(DS)\}$
(c) $\sim \alpha \to \sim\sim \alpha$ $\{(MP), (a), (b)\}$
(d) $(\sim \alpha \to \sim\sim \alpha) \to \sim\sim \alpha$ $\{(CMn))\}$
(f) $\sim\sim \alpha$ $\{(MP), (c), (d)\}$.

(CON).

(a) $\sim (\alpha \to \beta) \to \sim\sim\sim (\alpha \to \beta))$ by DT
(b) $\sim\sim (\alpha \to \beta)$ by DT
(c) $\sim\sim (\alpha \to \beta) \to (\sim\sim\sim (\alpha \to \beta) \to (\alpha \to \beta))$ $\{(DS)\}$
(d) $\sim\sim\sim (\alpha \to \beta) \to (\alpha \to \beta)$ $\{(MP), (b), (c)\}$
(e) $\sim (\alpha \to \beta) \to (\alpha \to \beta)$ $\{(MP), (TR), (a), (d)\}$
(f) $(\sim (\alpha \to \beta) \to (\alpha \to \beta)) \to (\alpha \to \beta)$ $\{(CMv) : \ \alpha \to \beta/\alpha\}$
(g) $\alpha \to \beta$ $\{(MP), (e), (f)\}$.

Now we can prove that (A3) and (A4) are theses of I^*.

(A3).

(a) $\sim\sim \alpha \to \sim \beta$ by DT
(b) $\sim\sim \alpha \to \beta$ by DT
(c) $(\beta \to (\sim \beta \to \sim \alpha)) \to (\sim\sim \alpha \to (\sim \beta \to \sim \alpha))$ $\{(MP), (TR), (b)\}$
(d) $\beta \to (\sim \beta \to \sim \alpha)$ $\{(DS)\}$
(e) $\sim\sim \alpha \to (\sim \beta \to \sim \alpha)$ $\{(MP), (d), (c)\}$
(f) $\sim \beta \to (\sim\sim \alpha \to \sim \alpha)$ $\{(MP), (PER), (e)\}$
(g) $((\sim \beta \to (\sim\sim \alpha \to \sim \alpha)) \to (\sim\sim \alpha \to$
$(\sim\sim \alpha \to \sim \alpha)))$ $\{(MP), (TR), (a)\}$
(h) $\sim\sim \alpha \to (\sim\sim \alpha \to \sim \alpha)$ $\{(MP), (f), (g)\}$
(i) $\sim\sim \alpha \to \sim \alpha$ $\{(MP), (SIM), (h)\}$
(j) $(\sim\sim \alpha \to \sim \alpha) \to \sim \alpha$ $\{(CMv) : \sim \alpha/\alpha\}$
(k) $\sim \alpha$ $\{(MP), (i), (j)\}$.

(A4).

(a) $\sim\sim (\alpha \to \beta)$ by DT
(b) $\sim (\alpha \to \beta) \to \sim\sim\sim (\alpha \to \beta)$ $\{(NN1)\}$
(c) $\sim\sim (\alpha \to \beta) \to (\alpha \to \beta)$ $\{(MP), (CON), (b)\}$
(d) $\alpha \to \beta$ $\{(MP), (a), (c)\}$.

□

It is an immediate consequence of Fact 2 that the calculus $I1$ is axiomatizable by (A1), (A2), (PL), (DS), (CMv) and (MP).

3. The Issue of Duality

Just like $I1$, the calculus $P1$ is expressed in a language using negation and implication as primitives. In this language $P1$ is axiomatized by

(A1) $\alpha \to (\beta \to \alpha)$
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
(A3) $(\sim \alpha \to \sim \beta) \to ((\sim \alpha \to \sim\sim \beta) \to \alpha)$
(A4) $\sim (\alpha \to \sim\sim \alpha) \to \alpha$
(A5) $(\alpha \to \beta) \to \sim\sim (\alpha \to \beta)$.

The sole rule of inference is (MP).

It is worth mentioning that (A4) is not independent (*cf.* [10, p. 155]).

FACT 3. The formulas

(DSv) $\alpha \to (\sim \alpha \to \beta)$, if $\alpha \notin var$
(CM) $(\sim \alpha \to \alpha) \to \alpha$
(NN1v) $\alpha \to \sim\sim \alpha$, if $\alpha \notin var$
(NN2) $\sim\sim \alpha \to \alpha$

are provable in $P1$.

FACT 4. (See [3]) $P1$ is axiomatizable by (A1), (A2), (PL), (DSv), (CM) and (MP).

The axiom (DSv) is of special interest here because it reveals that $P1$ behaves in a *paraconsistent* manner only at the level of variables, i.e. $\alpha \to (\sim \alpha \to \beta)$ is a $P1$-tautology only if α is not a propositional variable. Similarly to the calculus $I1$, the thought behind this was to demonstrate that it is possible to obtain a new set of axioms for $P1$ by imposing an additional condition on one of axioms given in Section 1. At this time, however, it is the axiom (DS). In this sense, the calculi $I1$ and $P1$ may be seen as *dual* (at least from the axiomatic perspective).

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