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Kripke Incompleteness of First-order Calculi with Temporal Modalities of CTL and Near Logics[1]

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We study an expressive power of temporal operators used in such logics of branching time as computational tree logic or alternating-time temporal logic. To do this we investigate calculi in the first-order language enriched with the temporal operators used in such logics. We show that the resulting languages are so powerful that many 'natural' calculi in the languages are not Kripke complete; for example, if a calculus in such language is correct with respect to the class of all serial linear Kripke frames (even just with constant domains) then it is not Kripke complete. Some near questions are discussed.

Keywords: Kripke incompleteness, first-order logic, computational tree logic, alternating-time temporal logic, recursive enumerability

1 Preliminaries

The subject under our consideration is an expressive power of temporal modalities used in such logics as **CTL***, **CTL**, **LTL**, **ATL***, **ATL**, etc., see [1, 7, 9, 18]. Here we dwell on the modalities of **CTL** but the argumentation below remains to be applicable for other logics, too (and we shall show this).

All the logics mentioned above are defined via Kripke semantics, and are Kripke complete by their definitions. It is known that they are decidable and even that the corresponding decision problems are complete in such classes as PSPACE (for **LTL**, see [21]), EXPTIME (for **CTL** and **ATL**, see [11, 24]), and 2-EXPTIME (for **CTL*** and **ATL***, see [12, 19, 23]). As a corollary of their decidability, they have decidable axiomatizations.

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But note that some modalities of these logics are not first-order definable (by means of appropriate first-order languages describing Kripke structures) and 'contain' an expressive power that may be not seen if we consider propositional languages only. Therefore, to show some possibilities of the modalities we add them to the first-order classical language and then propose and discuss some facts concerning logics and classes of logics in resulting languages.

Mathematical results presented here, in fact, follow from constructions used to prove that some first-order logics defined by classes of Kripke frames are not recursively enumerable. So the reader may see on this paper as a discussion on just one of corollaries from such proofs.

2 Decidability and recursive enumerability

Here we just recall the notions of decidability and recursive enumerability. Let U be some universal set (for our purposes it is sufficient U to be the set of all formulas in a certain language) and let X be a subset of U. Then X is called decidable if there exists an algorithm $\mathfrak{A}$ such that, for any $x \in U$,

$$\mathfrak{A}(x) \;=\; \begin{cases} 1, & \text{if } x \in X, \\ 0, & \text{if } x \notin X. \end{cases}$$

If X is not a decidable set then it is called undecidable. The set X is called recursive enumerable if $X = \varnothing$ or there exists an algorithm $\mathfrak{A}$ such that $X = \{\mathfrak{A}(n) : n \in \mathbb{N}\}$, i. e., there exists an algorithm enumerating elements in X. Note also that X is recursively enumerable if and only if there exists an algorithm $\mathfrak{A}$ such that, for any $x \in U$,

$$\mathfrak{A}(x) \;=\; \begin{cases} \text{something}, & \text{if } x \in X, \\ \text{not defined}, & \text{if } x \notin X, \end{cases}$$

i. e. X is the domain of the algorithm $\mathfrak{A}$. For more details see [8, 13, 20].

3 Calculi

Let us clarify what we mean by a calculus. Usually it is assumed that calculus is defined by a set of axioms and a set of inference rules. Both sets together, in fact, generate a set of derivable formulas. For our purposes it is important to be sure that such generation can be realized as an algorithmic procedure. Therefore we just add the following natural conditions: the set of axioms and the set of inference rules must be recursively enumerable and every inference rule must be realizable as an algorithm. The only property of calculi we are going to use is that the set of derivable formulas is recursively enumerable; it is ensured by the conditions.

Note, by the way, that any calculus with finite set of axioms and finite set of finitary inference rules, of course, satisfies both conditions above.

Below we sometimes equate a calculus to the set of all formulas derivable in it.

4 Language under consideration

Let us fix a language $\mathcal{L}$ containing a countable set of individual variables, a countable set of predicate letters of any arity (for every $m \in \mathbb{N}$, the language contains a countable set of m-ary predicate letters), $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), $\neg$ (negation), quantifiers on individual variables $\forall x$ and $\exists x$ (for every variable x), modalities $\boldsymbol{AX}$, $\boldsymbol{AF}$, $\boldsymbol{EU}$, and technical symbols (comma and parentheses). In other words, we enrich the classical first-order language with the modalities of **CTL**. Formulas are constructed in the usual way: if $x_1, \ldots, x_m$ are variables, P is m-ary predicate letter then $P(x_1, \ldots, x_m)$ is a formula; if φ and ψ are formulas and x is a variable then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $\neg\varphi$, $\forall x\, \varphi$, $\exists x\, \varphi$, $\boldsymbol{AX}\varphi$, $\boldsymbol{AF}\varphi$, and $(\varphi \boldsymbol{EU} \psi)$ are formulas, too.

5 Kripke semantics

By a Kripke frame here we understand a triple $\mathfrak{F} = \langle W, R, D\rangle$ where W is a non-empty set of states, R is a serial binary accessibility relation on W, and D is a function associating with every state s its domain (i.e., some non-empty set of individuals) such that $D(s) \subseteq D(t)$ whenever sRt, for any $s, t \in W$. Kripke model on a frame $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$where I is an interpretation of predicate letters in the domains of states, i.e., if P is n-ary predicate letter and s is a state then $I(s, P)$ is an n-ary relation on $D(s)$.

An infinite sequence $\pi = s_0, s_1, s_2, \ldots$ is called path in a frame $\mathfrak{F} = \langle W, R, D\rangle$ if, for any $k \in \mathbb{N}$, we have $s_k \in W$ and $s_k R s_{k+1}$. We assume that π_k denotes the k-th element of the path π. We say that a path π starts from a state s if $\pi_0 = s$. Note that, because R is serial, for any $s \in W$, there is at least one path in $\mathfrak{F}$ starting from s.

Let s be a state in a frame $\mathfrak{F} = \langle W, R, D\rangle$. A function α is called interpretation of individual variables in s if $\alpha(x_i) \in D(w)$, for every individual variable x_i.

Note that if s' is accessible from s and α is an interpretation of individual variables in s then α is an interpretation of individual variables in s', too, because in this case we have $D(w) \subseteq D(w')$.

For any individual variable x, let us define the binary relation $\overset{x}{=}$ on interpretations: for interpretations α and β we put

$$\alpha \overset{x}{=} \beta \quad \leftrightharpoons \quad \alpha(y) = \beta(y), \text{ for any variable } y \text{ such that } y \neq x.$$

Let $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ be a model on a serial frame $\mathfrak{F} = \langle W, R, D \rangle$. We define the truth relation 'a formula φ is true at a state $s \in W$ in a model $\mathfrak{M}$ under an interpretation α of individual variables in s' inductively (by constructing of φ). We put

$$(\mathfrak{M}, s) \models^{\alpha} P(x_1, \ldots, x_m) \quad \leftrightharpoons \quad \langle \alpha(x_1), \ldots, \alpha(x_m) \rangle \in I(s, P)$$

where P is m-ary predicate letter, $x_1, \ldots, x_m$ are individual variables. For other formulas the relation is defined as follows:

$(\mathfrak{M}, s) \models^{\alpha} \varphi_1 \wedge \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, s) \models^{\alpha} \varphi_1$ and $(\mathfrak{M}, s) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, s) \models^{\alpha} \varphi_1 \vee \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, s) \models^{\alpha} \varphi_1$ or $(\mathfrak{M}, s) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, s) \models^{\alpha} \varphi_1 \rightarrow \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, s) \not\models^{\alpha} \varphi_1$ or $(\mathfrak{M}, s) \models^{\alpha} \varphi_2$;

$(\mathfrak{M}, s) \models^{\alpha} \neg\varphi_1 \quad \leftrightharpoons \quad (\mathfrak{M}, s) \not\models^{\alpha} \varphi_1$;

$(\mathfrak{M}, s) \models^{\alpha} \boldsymbol{AX}\varphi_1 \quad \leftrightharpoons$ for any path π starting from s the relation $(\mathfrak{M}, \pi_1) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, s) \models^{\alpha} \boldsymbol{AF}\varphi_1 \quad \leftrightharpoons$ for any path π starting from s there is some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_1$;

$(\mathfrak{M}, s) \models^{\alpha} \varphi_1 \boldsymbol{EU} \varphi_2 \quad \leftrightharpoons$ for some path π starting in s and some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_2$ and, for any $j \in \mathbb{N}$, such that $j < k$ the relation $(\mathfrak{M}, \pi_j) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, s) \models^{\alpha} \forall x_i \varphi_1 \quad \leftrightharpoons$ for any interpretation β such that $\beta \overset{x_i}{=} \alpha$ and $\beta(x_i) \in D(s)$ the relation $(\mathfrak{M}, s) \models^{\beta} \varphi_1$ is true;

$(\mathfrak{M}, s) \models^{\alpha} \exists x_i \varphi_1$ $\leftrightharpoons$ there is an interpretation β such that $\beta \overset{x_i}{=} \alpha$, $\beta(x_i) \in D(s)$, and $(\mathfrak{M}, s) \models^{\beta} \varphi_1$.

As usual, a formula is said to be true in a model if it is true at any state in it; a formula is said to be true in a frame if it is true in any model on the frame; a formula is said to be true in a class of frames if it is true in any frame in the class.

Let us define the logic **QCTL** as the set of formulas that are true in the class of all (serial) Kripke frames.

6 Kripke completeness

We say that a set L of formulas is Kripke complete if there is a class of Kripke frames such that L coincides with the set of all formulas that are true in the class.

Note that if a set L is Kripke complete then it is closed, at least, under modus ponens, generalization, and predicate substitution, i.e., L may be viewed as a logic.

For example, **QCTL** is Kripke complete by its definition; any proper subset of **QCTL** is not Kripke complete (if we do not restrict the language and do not extend the class of frames, of course).

7 Logic QCTLlinCD

For some technical purposes we need to define a special extension of **QCTL**. We call a frame linear if the reflexive and transitive closure of its accessibility relation is linear. The frame $\langle W, R, D\rangle$ is said to be a frame with constant domains if $D(s) = D(t)$ whenever sRt, for any $s, t \in W$. Define **QCTLlinCD** as the set of formulas complete under the class of all linear frames with constant domains.

8 Class of Kripke incomplete calculi

Here we just propose and prove a statement which reflects the topic of the paper.

THEOREM 1. *Let S be a calculus such that $S \subseteq$ **QCTLlinCD**. Then S is not Kripke complete.*

PROOF. Let us denote by $\mathbf{QCL}_{fin}$ the classical theory of finite models. In [15] it is proved that there is a translation *Emb* such that, for any closed classical first-order formula φ,

$$\varphi \in \mathbf{QCL}_{fin} \iff Emb(\varphi) \in \mathbf{QCTL}.$$

More exactly, it is shown that, for any closed classical first-order formula φ,

$$\begin{array}{lcl}\varphi \in \mathbf{QCL}_{fin} & \Longrightarrow & \mathit{Emb}(\varphi) \in \mathbf{QCTL};\\ \varphi \notin \mathbf{QCL}_{fin} & \Longrightarrow & \mathit{Emb}(\varphi) \notin \mathbf{QCTLlinCD}.\end{array}$$

The second implication follows from the fact that if $\varphi \notin \mathbf{QCL}_{fin}$ then $\mathit{Emb}(\varphi)$ is refuted in a linear frame with constant domains (for details see [15]).

Because $\mathbf{QCL}_{fin}$ is not recursively enumerable [6], from these two implications it immediately follows that any set of formulas between **QCTL** and **QCTLlinCD** is not recursively enumerable, too. Indeed, let L be any set of formulas such that $\mathbf{QCTL} \subseteq L \subseteq \mathbf{QCTLlinCD}$. Let φ be a closed first-order formula. If $\varphi \in \mathbf{QCL}_{fin}$ then, by the first implication $\mathit{Emb}(\varphi) \in \mathbf{QCTL}$, and hence, $\mathit{Emb}(\varphi) \in L$; if $\varphi \notin \mathbf{QCL}_{fin}$ then, by the second implication, $\mathit{Emb}(\varphi) \notin \mathbf{QCTLlinCD}$, and hence, $\mathit{Emb}(\varphi) \notin L$. Therefore,

$$\varphi \in \mathbf{QCL}_{fin} \iff \mathit{Emb}(\varphi) \in L,$$

and, as a corollary, L is not recursively enumerable.

Suppose that S is Kripke complete. Then $\mathbf{QCTL} \subseteq S$. Together with the condition that $S \subseteq \mathbf{QCTLlinCD}$ it means that $\mathbf{QCTL} \subseteq S \subseteq \mathbf{QCTLlinCD}$, and hence it is not recursively enumerable. But this is impossible because S is a calculus. From the contradiction it follows that S is not Kripke complete. The theorem is proved. □

9 Discussion

Now we have got a matter for our discussion: the theorem and its proof. Both the theorem and the proof are quite short but we want to show some hidden details.

9.1 Examples of Kripke incomplete calculi

First of all, we give an explanation how to apply the theorem. Suppose we have some calculus S with a set of axioms $\mathcal{A}$ and a set of inference rules $\mathcal{R}$. Suppose also that we are able to check, for every $\varphi \in \mathcal{A}$, whether φ is true in the class of all linear frames with constant domains and, for every rule in $\mathcal{R}$, whether the rule is admissible in the same class (note, by the way, that there is no general procedure solving these tasks [15] but sometimes it is not the case for particular calculi). Then, if for every axiom and every rule the check is OK, then by the theorem we may conclude that S is not Kripke complete.

We give an example. Let $\mathcal{A} = \mathbf{CTL} \cup \mathbf{QCL}$, where **QCL** is the classical first-order logic, and let $\mathcal{R}$ be consisting of modus ponens, generalization, and substitution. Then the calculus defined by $\mathcal{A}$ and $\mathcal{R}$ is not Kripke complete.

If we extend the calculus with any formulas that are true in linear frames with constant domains (for example, bounded width formulas, bounded branching formulas, Barcan formula, etc.) and inference rules preserving validity in all such frames (for example, necessitation rule for the modality $\boldsymbol{AX}$) then we again obtain a Kripke incomplete calculus.

9.2 Possibility of constructive proofs for the theorem

Note that the proof presented here is not constructive: to prove the theorem we suppose it to be wrong and then obtain a contradiction. To give a constructive proof we must construct a formula φ that is not derivable in S but true in any Kripke frame for S.

Obviously, there is no such a formula for all calculi. Indeed, suppose φ is not derivable in any calculus S such that $S \subseteq \mathbf{QCTLlinCD}$ but $\varphi \in \mathbf{QCTLlinCD}$. If we add φ to S as an extra axiom then we obtain a calculus included into **QCTLlinCD** and containing φ, that gives us a contradiction.

Therefore, to get a constructive proof we need an effective procedure finding, for any calculus S such that $S \subseteq \mathbf{QCTLlinCD}$, a formula φ_S such that $\varphi_S \in \mathbf{QCTLlinCD}$ but φ_S is not derivable in S. The problem is in that the set of all such calculi (i. e., in fact, the set of inputs for the procedure) is not effectively definable because it is not recursively enumerable. Indeed, let $S_\varphi = \{\varphi\}$, for every formula φ (i. e., S_φ is a calculus with one axiom and without inference rules). Then the set $\{S_\varphi : S_\varphi \subseteq \mathbf{QCTLlinCD}\}$ coincides with the set $\{\{\varphi\} : \varphi \in \mathbf{QCTLlinCD}\}$ and hence it is not recursively enumerable. The same argumentation (with slight modifications) works also for calculus containing **QCL** and closed under some 'natural' inference rules; we leave details to the reader.

Nevertheless, of course, there are constructive ways to prove nearly the same theorem. Because any formula is constructed effectively, by the theorem we have that for any calculus S such that $S \subseteq \mathbf{QCTLlinCD}$ there is an algorithm constructing a formula φ_S such that $\varphi_S \in \mathbf{QCTLlinCD}$ but φ_S is not derivable in S. Therefore, for any particular calculus S there is a constructive proof of its Kripke incompleteness. Clearly, if $S' \subseteq S$ and φ is not derivable in S then φ is not derivable in S', too. Hence, we may replace **QCTLlinCD** in the theorem with any particular calculus and then obtain a constructive proof. For example, instead of **QCTLlinCD** we may take a calculus containing **QCL**, **CTL**, linearity axiom, Barcan formula and closed under substitution, modus ponens, generalization, and maybe some other inference rules.

We did not try to obtain a constructive proof this way, and here we leave the details of the question to the reader.

9.3 Extensions of the language

Note that **CTL** is a fragment of **CTL*** and, modulo some translation, it is also a fragment of **ATL** and **ATL***. It means that we can repeat our argumentation for logics **QCTL***, **QATL**, **QATL*** (the reader may define them using corresponding Kripke semantics for **CTL***, **ATL**, **ATL***). But, in fact, we do not need it: it is enough to use the theorem for calculi in extended language. Let us understand $\mathcal{L}$-fragment of a calculus (in a language extending $\mathcal{L}$) as a set consisting of all formulas derivable in the calculus that are in $\mathcal{L}$ (maybe, modulo a certain translation).

Corollary 1. *Let S be a calculus in the language of* **QCTL****,* **QATL** *or* **QATL*** *such that the $\mathcal{L}$-fragment of S is a subset of* **QCTLlinCD**. *Then S is not Kripke complete.*

Moreover, we may imagine a situation when we deal with some different language allowing to express $\mathcal{L}$ inside of it. Then the corollary is true, too.

9.4 Fragments of the language

Now let us turn to another ‘direction’, and put the following question: what happens if we restrict $\mathcal{L}$?

Due to S. Kripke [16], if we restrict $\mathcal{L}$ with just unary predicate letters then both the theorem and the corollary are still true; moreover, we propose a hypothesis that sometimes even one unary letter is enough [3, 4]. As for individual variables, we think that three ones are enough; maybe even two [14]. But here we discuss the modalities, therefore we consider some restrictions on their using.

In accordance with literature on **CTL**, we distinguish five ‘basic’ modalities: $\boldsymbol{AX}$, $\boldsymbol{AG}$, $\boldsymbol{AF}$, $\boldsymbol{EU}$, and $\boldsymbol{AU}$. Formally, the language $\mathcal{L}$ already contains $\boldsymbol{AX}$, $\boldsymbol{AF}$, and $\boldsymbol{EU}$, therefore we define just $\boldsymbol{AG}$ and $\boldsymbol{AU}$: $\boldsymbol{AG}\varphi = \neg\boldsymbol{E}((\varphi \to \varphi)\boldsymbol{U}\neg\varphi)$, $\boldsymbol{A}(\varphi\boldsymbol{U}\psi) = \boldsymbol{AF}\varphi \wedge \neg\boldsymbol{E}(\neg\psi\boldsymbol{U}(\neg\varphi \wedge \neg\psi))$. Note that we also may define five dual modalities (known as $\boldsymbol{EX}$, $\boldsymbol{EF}$, $\boldsymbol{EG}$, $\boldsymbol{AR}$, and $\boldsymbol{ER}$) but they are not essential for our purposes, and we leave details to the reader [1, 9, 17]. Every subset of the set $\{\boldsymbol{AX}, \boldsymbol{AG}, \boldsymbol{AF}, \boldsymbol{EU}, \boldsymbol{AU}\}$ defines a certain fragment of **QCTL**, and we consider such fragments.

Let M be a subset of $\{\boldsymbol{AX}, \boldsymbol{AG}, \boldsymbol{AF}, \boldsymbol{EU}, \boldsymbol{AU}\}$. For a set L of formulas, define $L{\restriction}M$ as a fragment of L where only modalities contained in M are used. For example, $\mathbf{QCTL}{\restriction}\varnothing = \mathbf{QCL}$. In fact, in [15] it is proved that from $\mathbf{QCTL} \subseteq L \subseteq \mathbf{QCTLlinCD}$, $M \neq \varnothing$, $M \neq \{\boldsymbol{AX}\}$, and $M \neq \{\boldsymbol{AG}\}$ it follows that $L{\restriction}M$ is not recursively enumerable. Hence, for such fragments we may use the same argumentation as in the proof of the theorem. As a result we obtain the following proposition.

Proposition 1. *Let M be a set of modalities allowing to express at least one of the modalities $\boldsymbol{AF}$, $\boldsymbol{EU}$, $\boldsymbol{AU}$ or both $\boldsymbol{AX}$ and $\boldsymbol{AG}$, let also S be a calculus such that $S{\restriction}M \subseteq \mathbf{QCTLlinCD}{\restriction}M$; then S is not Kripke complete.*

This proposition is stronger than the theorem. It shows that just one temporal modality may be quite expressive. But note that the proposition does not tell us anything about calculi in the language with $\boldsymbol{AX}$ only and with $\boldsymbol{AG}$ only.

9.5 Effect of first-order conditions

To define Kripke semantics for $\mathcal{L}$ we need the notion of path. Formally, π is a path in a frame $\langle W, R, D\rangle$ if π is a map from $\mathbb{N}$ to W such that $\pi(n)R\pi(n+1)$, for every $n \in \mathbb{N}$. Then, to define the truth relation for the modalities, in fact, we use second-order quantifiers (on paths).

It is not the case for $\boldsymbol{AX}$. It is possible to define the truth relation for it using just R and first-order quantifiers (on states in W): $\boldsymbol{AX}\varphi$ is true at a state s if φ is true at every state t such that sRt. Together with first-order definability of seriality and linearity it provides us with embeddings of $\mathbf{QCTL}{\restriction}\{\boldsymbol{AX}\}$ and $\mathbf{QCTLlinCD}{\restriction}\{\boldsymbol{AX}\}$ into $\mathbf{QCL}$, and hence with a recursive axiomatization for each of them. Of course, it is not so for any logic between $\mathbf{QCTL}{\restriction}\{\boldsymbol{AX}\}$ and $\mathbf{QCTLlinCD}{\restriction}\{\boldsymbol{AX}\}$ into $\mathbf{QCL}$ but from the construction in [5] we obtain the following observation: *let a logic L be Kripke complete under some first-order definable class of frames; then $L{\restriction}\{\boldsymbol{AX}\}$ is recursively enumerable.* Note that any recursively enumerable logic has also a recursive axiomatization [10]. Note also that we do not know whether the converse statement for the observation holds.

To define the truth relation for $\boldsymbol{AG}$, in fact, we must define reflexive and transitive closure of arbitrary accessibility relation R. It is not possible to define it via R and equality if we use the first-order language only, but note that if we deal with the modality $\boldsymbol{AG}$ without all others, then we may 'forget' about R and use its reflexive and transitive closure as a unique binary relation in a frame. In this case, we must just claim it to be reflexive and transitive. Then, we are again in the similar situation: there are embeddings of $\mathbf{QCTL}{\upharpoonright}\{\boldsymbol{AG}\}$ and $\mathbf{QCTLlinCD}{\upharpoonright}\{\boldsymbol{AG}\}$ into **QCL**, wherefore these fragments (and some ones between them) are recursively axiomatizable.

Of course, if we consider classes of frames allowing us to define other modalities using first-order conditions only, then we obtain recursively axiomatizable extensions of **QCTL**. For example, if a logic L is complete under a class consisting of all frames $\langle W, R, D\rangle$ with the same finite W and the same R on it then L is recursively axiomatizable.

9.6 Other classes

Our main conclusions and observations are based on the fact that any set between **QCTL** and **QCTLlinCD** is not recursively enumerable. What about logics outside the interval? In general, we do not know. But we show some difficulties. To do this, consider an example.

Let **QCTLfin** be the logic complete under the class of all finite Kripke frames. This logic is not included into **QCTLlinCD** but using argumentation as in [22] we obtain that **QCTLfin** is not recursively enumerable (and even $\mathbf{QCTLfin}{\upharpoonright}\{\boldsymbol{AX}\}$ is not recursively enumerable). To prove the theorem as above but with **QCTLfin** instead of **QCTLlinCD** we need an algorithm embedding some non-enumerable problem into both **QCTL** and **QCTLfin**, simultaneously. But we do not know whether such an algorithm exists.

Let **QCTLfinCD** be the logic complete under the class of all finite Kripke frames with constant domains. It is not recursively enumerable, too. Moreover, it is possible to show that there is a translation tr such that, for any closed classical first-order formula φ,

$$\begin{array}{rcl}\varphi \in \mathbf{QCTLfin} & \Longleftrightarrow & tr(\varphi) \in \mathbf{QCTLfin}\\ & \Longleftrightarrow & tr(\varphi) \in \mathbf{QCTLfinCD},\end{array}$$

and hence, any set of formulas between **QCTLfin** and **QCTLfinCD** is not recursively enumerable (unfortunately, we do not know about any publications containing this fact, and cannot give a reference). It seems we are successful and can extend the class of Kripke incomplete calculi. But this is not so, again. Indeed, in this case, we just may propose that any calculus such that $\mathbf{QCTLfin} \subseteq S \subseteq \mathbf{QCTLfinCD}$ is not Kripke complete but, clearly, there is no such calculus.

9.7 Impossibility of 'converse' embeddings

In fact, our proof of the theorem is based on the fact that there exists an embedding of $\mathbf{QCL}_{fin}$ into any theory between **QCTL** and **QCTLlinCD**. The following natural question arises: is there an effective embedding of **QCTL** or **QCTLlinCD** into $\mathbf{QCL}_{fin}$? The answer is 'no', and here we give some argumentation.

Let us recall Post theorem, see [2]:

- *a set X is decidable if and only if both X and $\overline{X}$ are recursively enumerable,*

where $\overline{X}$ is the complement of X. Clearly, for a logic L and a formula φ, we have that

- φ is L-valid if and only if $\neg\varphi$ is not L-satisfiable;
- φ is not L-satisfiable if and only if $\neg\varphi$ is L-valid.

Therefore, in terms of L-validity and L-satisfiability, Post theorem means that

- *L is decidable if and only if both L-validity problem and L-satisfiability problem are recursively enumerable,*

and hence, if L is undecidable then at least one of the problems is not recursively enumerable.

The following statement is known as Church theorem, see [8, 13]: **QCL** is not decidable. But because **QCL** is finitely axiomatizable, it is also recursively enumerable. Therefore, using Post theorem, we may conclude that

- **QCL**-validity problem is recursively enumerable;
- **QCL**-satisfiability problem is not recursively enumerable.

Note also that the set of all $\mathbf{QCL}_{fin}$-satisfiable formulas is recursively enumerable: corresponding algorithm just tests for a given formula whether it is true in models with one element, then whether it is true in models with two elements, then whether it is true in models with three elements, and so on; if the algorithm finds a model satisfying the formula then it stops with the positive answer. Therefore, we may specify Trakhtenbrot theorem [6]:

- $\mathbf{QCL}_{fin}$-validity problem is not recursively enumerable;

- $\mathbf{QCL}_{fin}$-satisfiability problem is recursively enumerable.

Observe that for any closed classical first-order formula φ,

$$\begin{aligned}\varphi \in \mathbf{QCL} &\iff \varphi \in \mathbf{QCTL}\\ &\iff \varphi \in \mathbf{QCTLlinCD}.\end{aligned}$$

Because **QCL**-satisfiability problem is not recursively enumerable, as a corollary we obtain that L-satisfiability problem is not recursively enumerable, too, for any logic between **QCTL** and **QCTLlinCD** (of course, the same is also true for any logic between **QATL** and **QATLlinCD**, etc.).

Let L be a logic between **QCTL** and **QCTLlinCD**. Suppose that there exists an embedding of L into $\mathbf{QCL}_{fin}$, i. e., there exists an algorithm A such that

$$\varphi \in L \iff A(\varphi) \in \mathbf{QCL}_{fin},$$

for any formula φ in the language $\mathcal{L}$. Then we immediately obtain that $\mathbf{QCL}_{fin}$-satisfiability problem is not recursively enumerable but it is not so. The contradiction means that there is no such embedding. The same argumentation allows us to conclude that there is no embedding of L into **QCL**, too.

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