Cardinality of sets of closed functional classes in weak 3-valued logics

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abstract. This paper proves that sets of closed functional classes in 3-valued logics of Bochvar B_3 and Hallden H_3 contains a continuum of different closed classes. It is also proven that both of these logics contain a closed functional class which has no basis.

Keywords: Bochvar's logic, Hallden's logic, closed class, continuum, cardinality

The research on cardinality of closed sets of functions in different logics was started by E. Post. Thus, in [10] he proved that classical logic only contains an enumerable set of different closed functional classes. In 1959 Yu.I. Yanov and A.A. Muchnik [3] for the first time showed that for every $k \geq 3$ the k-valued Post's logic P_k contains a closed class which has no basis, and also contains a continual set of different closed functional classes. M.F. Ratsa in [4] and [5] showed that 3-valued logic of Heyting G_3 contains a continual set of different functional classes which have bases and a continual set of classes which have no bases. Consequently, cardinalities of sets of closed functional classes in different logics were researched in the fundamental monograph [9] by D. Lau which deals with functional algebras on finite sets.

A.S. Karpenko in [2] suggested a hypothesis that the set of closed classes in Bochvar's 3-valued logic B_3 has the power of continuum (truth-tables, defining basic functions of B_3, will be formulated below). As a justification of this hypothesis the author uses the condition (see [9, pp. 221–222]) for a class to contain just an enumerable set of closed functional classes. A.S. Karpenko found out that logic B_3 does not satisfy this criterion. Nevertheless, this condition is a sufficient but not necessary one. Despite this argument, we cannot conclude that the set of closed classes in B_3 is continual.

Thus, the question on cardinality of the set of closed classes in B_3 remained open until now. It was also unknown, whether there are functional classes in B_3 (id est logics weaker, than B_3 itself) which contain continual sets of closed classes. In this paper the author gives positive answers to these questions, and the answer to the latter may also be viewed as an answer to the former of them. In particular, as 3-valued Hallden's logic H_3 which was first studied in [8] contains (as shown below) a continual set of different closed classes, and as H_3 is precomplete in B_3 (this fact was proven by V.K. Finn in [6]), the set of closed classes in B_3 is continual.

Let us formulate a series of corresponding theorems and prove them. For this purpose we shall use the slightly modified strategy of Yu.I. Yanov and A.A. Muchnik.

Below we shall use basic functions of B_3, defined by the following truth-tables:

$\cap$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$\cup$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\sim$	x
0	1
$\frac{1}{2}$	$\frac{1}{2}$
1	0

$\vdash$	x
1	1
0	$\frac{1}{2}$
0	0

THEOREM 1. *The set of closed classes in B_3 contains a class, which has no basis.*

PROOF. Let us consider a sequence

$$S = f_0, f_1(x_1), f_2(x_1, x_2), ...$$

of functions $f_i(x_1, ..., x_i)$ of 3-valued Post logic P_3 for $i \in \{0, 1, 2, ...\}$, satisfying the following conditions:

$$f_0 \equiv 0;$$

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{if } x_1 = ... = x_i = \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

First of all, it is necessary to demonstrate that all the functions $f_i(x_1, ..., x_i)$ are in B_3. For this purpose it is sufficient to observe that the constant 0 is in B_3 and may be expressed with, for example, a formula $\vdash x \cap \sim\vdash x$, and to express the rest of the functions $f_i(x_1, ..., x_i)$ for every $i > 0$ with formulas:

$$\sim\vdash x_1 \cap \sim\vdash\sim x_1 \cap ... \cap \sim\vdash x_i \cap \sim\vdash\sim x_i.$$

It is worth mentioning that the formula $\sim\vdash x \cap \sim\vdash\sim x$ represents the Rosser–Turquette operator (J-operator) for the value $\frac{1}{2}$ in B_3[1]. So, the formula, expressing functions $f_i(x_1, ..., x_i)$, may be simplified with the use of J-operator for the value $\frac{1}{2}$ just as follows:

$$J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_i).$$

Let $\mathfrak{M}(S)$ be a class generated by the set of functions

$$\{f_0, f_1(x_1), f_2(x_1, x_2), ...\} \subset B_3$$

by renaming variables without identifying them. This class is a closed one. Let us also assume that $\mathfrak{M}(S)$ has a basis. In this case, there is a function f' that is obtained from function $f_{n_0}(x_1, ..., x_{n_0})$ through renaming variables for which the number n_0 is minimal. Then we have two cases:

1. The basis contains at least one more function f'' corresponding to a function $f_{n_1}(x_1, ..., x_{n_1})$ with $n_1 > n_0$. As $f_{n_0}(x_1, ..., x_{n_0})$ may be obtained from $f_{n_1}(x_1, ..., x_{n_1})$ by identifying some of the variables $x_1, ..., x_{n_1}$, the function f' may be expressed through f'', and this contradicts to the definition of a basis.

2. The basis consists of a single function f'. In this case no other function f_n for $n > n_0$ can be expressed with f', as $f_{n_0}(..., f_{n_0}, ...) \equiv 0$, that leads to a contradiction again. □

THEOREM 2. *There is a closed class with an enumerable basis in B_3.*

PROOF. To prove the theorem we shall consider a sequence

$$S = f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...$$

[1]RosserTurquette operators $J_1(x) = \vdash x$, $J_{\frac{1}{2}}(x) = \sim\vdash x \cap \sim\vdash\sim x$ and $J_0(x) = \vdash\sim x$ for B_3 were for the first time constructed by V.K. Finn in [7].

of functions $f_i(x_1, ..., x_i)$ in 3-valued logic of Post P_3 for $i \in \{2, 3, ...\}$, which satisfy the following conditions:

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{for } x_1 = ... = x_{j-1} = x_{j+1} = ... = x_i = \frac{1}{2}, \\ & x_j = 1 \ (1 \leq j \leq i); \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that for every i these functions can be defined using the basic functions of B_3. With this purpose for every x_j $(1 \leq j \leq i)$ and every i we shall consider the following formulae of B_3:

$F_j = \vdash x_j \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1) \cap ... \cap (\sim\vdash x_{j-1} \cap \sim\vdash\sim x_{j-1}) \cap (\sim\vdash x_{j+1} \cap \sim\vdash\sim x_{j+1}) \cap ... \cap (\sim\vdash x_i \cap \sim\vdash\sim x_i)$.

Then let F be the internal disjunction of formulae F_j:

$$F = \bigcup_1^i F_j.$$

For every fixed i, formulae F define functions $f_i(x_1, ..., x_i)$ from B_3. Thus, for example, for $i = 2$, there are only two formulae F_j, id est: $F_1 = \vdash x_1 \cap (\sim\vdash x_2 \cap \sim\vdash\sim x_2)$ and $F_2 = \vdash x_2 \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1)$. Then $F = F_1 \cup F_2$ is expressed in the following manner:

$(\vdash x_1 \cap (\sim\vdash x_2 \cap \sim\vdash\sim x_2)) \cup (\vdash x_2 \cap (\sim\vdash x_1 \cap \sim\vdash\sim x_1))$.

It is easy to verify that the function $f_2(x_1, x_2) \in B_3$ corresponding to this formula returns the value 1 only on two tuples $\langle 1, \frac{1}{2} \rangle$ and $\langle \frac{1}{2}, 1 \rangle$ of truth-values of variables x_1 and x_2. On all other tuples of truth-values this function returns the value 0.

Thus, it is proven that for every i functions $f_i(x_1, ..., x_i)$ are in B_3.

Notation of formulae F_j and F may be simplified essentially if we use J-operators for the truth-values 1 and $\frac{1}{2}$:

$$F'_j = J_1(x_j) \cap J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_{j-1}) \cap J_{\frac{1}{2}}(x_{j+1}) \cap ... \cap J_{\frac{1}{2}}(x_i).$$

Formula F, in this case, should be rewritten as the internal disjunction of all of the F'_j:

$$F' = \bigcup_1^i F'_j.$$

Let $\mathfrak{M}(S)$ be a closed class generated by the system of functions $\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$. We shall prove that this system is a basis for $\mathfrak{M}(S)$. It is sufficient to show that none of the functions $f_m(x_1, ..., x_m)$ in this class can be expressed only with functions

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\},$$

id est there is no representation:

$$f_m(x_1, ..., x_m) = \mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...].$$

The formula $\mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...]$ may be rewritten as:

$$\mathfrak{A}[f_2, ..., f_{m-1}, f_{m+1}, ...] =$$
$$= f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]).$$

Using the first equation, we have:

$$f_m(x_1, ..., x_m) =$$
$$= f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]).$$

Let us observe three possible cases:

1. At least two of the formulae among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...],$$

where $r \geq 2$, do not coincide with symbols of variables. In this case, for every m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values of variables $x_1, ..., x_m$, there are values 1 or 0 on corresponding argument places of the function

$$f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]),$$

and this function, according to its definition, will be equivalent to 0. That is a contradiction to the hypothesis that the function $f_m(x_1, ..., x_m)$ may be expressed only with functions from

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\},$$

as no function in the set

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$$

is equivalent to 0.

2. Only one formula $\mathfrak{B}_s$ among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]$$

does not coincide with a symbol of variable. In this case, functions corresponding to the rest of the formulae in this list are equivalent to variables, and, as $r \geq 2$, there is at least one formula $\mathfrak{B}_p \equiv x_q$. Let us consider an m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ such that $\alpha_1 = ... = \alpha_{q-1} = \alpha_{q+1} = ... = \alpha_m = \frac{1}{2}$, and $\alpha_q = 1$. On this ordered set of truth-values the function corresponding to the formula $\mathfrak{B}_s$ returns the value 1 or 0. Then on the m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ in the function

$$f_r(\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...])$$

there are at least two argument places having truth-values which do not coincide with $\frac{1}{2}$. Therefore, the right part of the equation is equal to 0, and its left part must, according to definition of the function $f_m(x_1, ..., x_m)$, be equal to 1 which is impossible.

3. All of the formulae among

$$\mathfrak{B}_1[f_2, ..., f_{m-1}, f_{m+1}, ...], ..., \mathfrak{B}_r[f_2, ..., f_{m-1}, f_{m+1}, ...]$$

are equivalent to symbols of variables. Then $r > m$, and there are at least two entries of some variable x_p in the formula expressing the function $f_m(x_1, ..., x_m)$. Considering the ordered m-tuple $\langle \alpha_1, ..., \alpha_m \rangle$ of truth-values for variables $x_1, ..., x_m$ such that $\alpha_1 = ... = \alpha_{p-1} = \alpha_{p+1} = ... = \alpha_m = \frac{1}{2}$ and $\alpha_p = 1$, we find out again that the right part of the corresponding equation is equal to 0, and its left part is equal to 1 which is impossible.

All three cases lead to contradiction. Therefore, none of the functions $f_m(x_1, ..., x_m)$ where $m \geq 2$ can be represented as a formula using only functions from

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\} \backslash \{f_m(x_1, ..., x_m)\}.$$

□

This theorem allows to prove one of the main results of this paper, that the set of closed classes in B_3 has the cardinality of continuum. The method for proving this result is the same with the strategy used by Yu.I. Yanov and A.A Muchnik, to prove continuality of the set of closed classes of functions in k-valued logics of Post P_k, for all $k \geq 3$.

THEOREM 3. *The class of functions of B_3 contains a continuum of different closed sets.*

PROOF. The upper bound for cardinality of the set of closed classes in B_3 coincides with cardinality of the set of all subsets of functions in B_3. As the set of functions in B_3 is enumerably infinite, the set of all subsets of this set has the cardinality of continuum.

To obtain the lower bound for cardinality of the set of closed classes in B_3 it is enough to consider the closed class $\mathfrak{M}(S)$ constructed in the previous theorem. This class has a basis

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}.$$

For every sequence $S' = s_1, s_2, ...$ of natural numbers, where $2 \leq s_1 < s_2 < ...$, let us consider a closed class $\mathfrak{M}(S')$ which has a following set of functions as its basis:

$$\{f_{s_1}(x_1, ..., x_{s_1}), f_{s_2}(x_1, ..., x_{s_2}), ...\}.$$

It is obvious that

$$\mathfrak{M}(s_1, s_2, ...) \neq \mathfrak{M}(s'_1, s'_2, ...),$$

if $\{s_1, s_2, ...\} \neq \{s'_1, s'_2, ...\}$.

Consequently, the set of closed classes $\{\mathfrak{M}(S')\}$ in the set of closed classes of B_3 is continual. □

A question arises, whether existence of a set

$$\{f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...\}$$

of functions defined in the previous manner in a functional class of some 3-valued logic is necessary for this logic to contain a continuum of different closed classes. In general, the answer to this question is 'wrong', as the above-formulated definition of the sequence S of functions $f_i(x_1, ..., x_i)$ $(i \geq 2)$ is, in a certain sense, too strong, because it requires possibility to use at least two J-operators, as we can do in B_3. But prerequisites of this definition may be weakened essentially, so that we shall be able to prove one of the key theorems of this paper about continuality of the set of closed classes in 3-valued logic of Hallden H_3.

The basic connectives of logic H_3 are those in the set $\{\sim, J_{\frac{1}{2}}, \cap, \cup\}$ (for example, see [1, p. 57]).

THEOREM 4. *The set of closed functional classes in H_3 contains a class, which has no basis.*

PROOF. The sequence of functions

$$S = f_0, f_1(x_1), f_2(x_1, x_2), ...$$

is defined, using Rosser–Turquette operator $J_{\frac{1}{2}}(x)$, just as it was done in Theorem 1. The rest of the proof is completely analogous to the proof of Theorem 1. $\square$

THEOREM 5. *The class of functions in H_3 contains a closed class, which has an enumerable basis.*

PROOF. To prove the theorem, consider a sequence

$$S = f_2(x_1, x_2), f_3(x_1, x_2, x_3), ...$$

of functions $f_i(x_1, ..., x_i)$ of 3-valued logic of Post P_3, for $i \in \{2, 3, ...\}$, satisfying the following definition:

$$f_i(x_1, ..., x_i) = \begin{cases} 1 & \text{if } x_1 = ... = x_{j-1} = x_{j+1} = ... = x_i = \frac{1}{2}, \\ & x_j \in \{1, 0\} \ (1 \leq j \leq i); \\ 0 & \text{otherwise.} \end{cases}$$

Let us show that such functions may be defined using the basic functions of H_3, for each i. With this purpose we need to consider, for every x_j $(1 \leq j \leq i)$ and every i, the following formulae of H_3:

$F_j = \sim J_{\frac{1}{2}}(x_j) \cap J_{\frac{1}{2}}(x_1) \cap ... \cap J_{\frac{1}{2}}(x_{j-1}) \cap J_{\frac{1}{2}}(x_{j+1}) \cap ... \cap J_{\frac{1}{2}}(x_i)$.

Then, for every i, let F be the internal disjunction of all of the formulae F_j:

$$F = \bigcup_{1}^{i} F_j.$$

The rest of the proof is analogous to the corresponding proof for B_3. □

Theorem 6. *There is a continuum of different closed functional classes among functions of* H_3.

Proof. The proof of this theorem is identical with the corresponding proof for B_3. □

It is worth mentioning that proofs of theorems 4–6 do not depend on proofs of theorems 1–3, and, as H_3 is precomplete in B_3, the former of them may be viewed as independent proofs of corresponding facts for B_3.

After proving these theorems one can suppose that enjoying the property of having a continuum of different closed functional classes for various multi-valued logics is rather a normal phenomenon, than a strange deviation, even for very weak multi-valued functional systems like H_3. If this hypothesis is true, it may be viewed as a new philosophical argument enforcing the thesis about qualitative difference between multi-valued logics and classical bivalent logic.

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