First-order logics of branching time: on expressive power of temporal operators[1]

Ekaterina Kotikova, Mikhail Rybakov

abstract. We consider the logic **QCTL**, a first-order extension of **CTL** defined as a logic of Kripke frames for **CTL**. We study the question about recursive enumerability of its fragments specified by a set of temporal modalities we use. Then we discuss some questions concerned axiomatizability and Kripke completeness.

Keywords: non-classical logic, temporal logic, branching time logic, first-order logic, recursive enumerability, Kripke completeness

1 Introduction

There are at least two reasons to study branching time logics: philosophical and originating in computer science. Such logics provides us with formalisms allowing to construct and verify sentences about indeterminate future (philosophical aspect) or about some state transition systems (in computer science).

There are a lot of *propositional* temporal logics, and they found their applications both in philosophy, and computer science, see [4]. Here we deal with *first-order* logics of branching time, more exactly, first-order extensions of the logic **CTL** introduced by A. Prior, see [10].[2] It is known that such logics are undecidable and even not

[1]The work was supported by Russian Foundation for Basic Research, Projects № 11–06–00456 and № 13–06–00861.

[2]The abbreviation 'CTL' means 'computational tree logic'. In [10] A. Prior does not introduce this abbreviation but he discuss logics of branching time, in particular, Cocchiarella's tense-logic (which may have the same abbreviation).

recursively enumerable; moreover for correspondent proofs it is sufficient to use only unary predicate letters, see [5, 6, 12].

The main aim of the paper is to show how one may prove that a logic (a fragment of a logic) is not recursively enumerable.

To do this we simulate positive integers with the relation 'less than'; this is the key part of the paper. Then, we use positive integers to embed the finite model theory (which is not recursively enumerable) into first-order branching time logic.

On the one hand, as a result we obtain that many fragments of logics we consider are not finitely (and even recursively) axiomatizable. Note that we define these logics semantically by means of Kripke frames; therefore, on the other hand, it follows that many calculi are not Kripke complete.

Note that the results (as theorems) presented in this paper are quite expected; moreover, most of them are known or follow from other known facts. The feature of our proofs is that, in fact, we use only embeddings of logics and nothing more. Therefore, to understand our proofs it is sufficient to be familiar with the classical first-order logic (theory) of finite domains.

2 Definitions

To define the logic we deal with, first of all we need a language. Consider the language containing

- individual variables $x_0, x_1, x_2, \ldots$;
- predicate letters P_i^m, for every $m, i \in \mathbb{N}$;
- logical constant $\bot$;
- logical connectives $\wedge$, $\vee$, $\rightarrow$;
- quantifier symbols $\forall$, $\exists$;
- modality symbols $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, $\boldsymbol{U}$;
- symbols (,), and comma.

Now define formulas we consider here. Atomic formulas are $\bot$ and $P_i^m(x_{k_1}, \ldots, x_{k_m})$ where $m, i, k_1, \ldots, k_m$ are positive integers.

If φ and ψ are formulas then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $\forall x_i \varphi$, $\exists x_i \varphi$, $\boldsymbol{AX}\varphi$, $\boldsymbol{EX}\varphi$, $\boldsymbol{AF}\varphi$, $\boldsymbol{EF}\varphi$, $\boldsymbol{AG}\varphi$, $\boldsymbol{EG}\varphi$, $(\varphi \boldsymbol{AU} \psi)$, and $(\varphi \boldsymbol{EU} \psi)$ are formulas, too. We call such formulas *temporal*.

Also we use $\top$, $\neg$, and $\leftrightarrow$ as usual abbreviations:

$$
\begin{array}{lcl}
\neg\varphi & = & (\varphi \to \bot); \\
\top & = & \neg\bot; \\
(\varphi \leftrightarrow \psi) & = & ((\varphi \to \psi) \wedge (\psi \to \varphi)).
\end{array}
$$

We omit parenthesis that can be recovered according to the following priority of the connectives: unary modalities, quantifiers, $\neg$, binary modalities, $\wedge$, $\vee$, $\leftrightarrow$, $\to$.

We make a remark about modalities used in formulas. Any 'atomic' modality consists of two symbols: the first symbol is $\boldsymbol{E}$ or $\boldsymbol{A}$ and the second one is $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, or $\boldsymbol{U}$. Every of these symbols corresponds to some modality in more general language, see [2]. The intending meaning of the modalities $\boldsymbol{E}$, $\boldsymbol{A}$, $\boldsymbol{X}$, $\boldsymbol{G}$, $\boldsymbol{F}$, $\boldsymbol{U}$ is as follows: let us imagine that we are in some situation (current state) and it is possible to define consequences of future states; then

$\boldsymbol{E}$	means	'there is a consequence of future states (starting in the current one) such that…';
$\boldsymbol{A}$	means	'for every consequence of future states (starting in the current one) it is true that…';
$\boldsymbol{X}\varphi$	means	'φ is true in the next state of the consequence';
$\boldsymbol{G}\varphi$	means	'φ is true in every state of the consequence';
$\boldsymbol{F}\varphi$	means	'φ is true in some state of the consequence';
$\varphi\boldsymbol{U}\psi$	means	'ψ is true in at least one state of the consequence and φ is true until ψ is true'.

Of course, since we use modalities only in pairs we have no formulas like $\boldsymbol{X}\varphi$ or $\varphi\boldsymbol{U}\psi$; but now we have some informal definition for our modalities. For example $\boldsymbol{AF}\varphi$ means that for every consequence of future states (starting in the current one), φ is true in some state of the consequence.

To make the meaning of the modalities more clear we need semantics. As semantics for this language we use Kripke frames and models.

A pair $\mathfrak{F} = \langle W, R\rangle$ is called *Kripke frame* if W is non-empty set and R is a binary relation on W. We call elements of W *worlds* or *states*; we call R *accessibility relation* on W. We write wRw' instead of $\langle w, w'\rangle \in R$; if wRw' we say that w' is *accessible* from w.

We may understand a Kripke frame as a structure of (branching) time where wRw' means that w' is a possible next future state relative to w.

Here we consider mainly serial Kripke frames; recall that a frame $\mathfrak{F} = \langle W, R\rangle$ is said to be *serial* if, for any $w \in W$, there is $w' \in W$ such that wRw'.

An infinite consequence $\pi = w_0, w_1, w_2, \ldots$ is called a *path* in a frame $\mathfrak{F} = \langle W, R\rangle$ if, for any $k \in \mathbb{N}$, we have $w_k \in W$ and $w_k R w_{k+1}$. We assume that π_k denotes the k-th element of the path π. We say that a path π *starts in a world* w if $\pi_0 = w$.

Note that if w is a world of a serial Kripke frame $\mathfrak{F}$ then there is at least one path in $\mathfrak{F}$ starting in w.

A triple $\mathfrak{F}(D) = \langle W, R, D\rangle$ is called *predicate Kripke frame* if $\langle W, R\rangle$ is a Kripke frame and D is a map associating with every $w \in W$ some non-empty set D_w (i. e., $D(w) = D_w$) such that

$$wRw' \implies D(w) \subseteq D(w'),$$

for any $w, w' \in W$. Elements in D_w are called *individuals* of the world w, the set $D(w)$ is called *domain* of w.

Now we need a tool connecting predicate frames with our language. As such tool we use two notions: interpretation of predicate letters and interpretation of individual variables.

Let $\mathfrak{F}(D) = \langle W, R, D\rangle$ be a predicate Kripke frame. A function I is called *interpretation of predicate letters* in $\mathfrak{F}(D)$ if $I(w, P_i^m)$ is an m-ary relation on $D(w)$, for every $w \in W$ and every predicate letter P_i^m.

A tuple $\mathfrak{M} = \langle W, R, D, I\rangle$ is called *Kripke model* if $\langle W, R, D\rangle$ is a predicate Kripke frame and I is an interpretation of predicate letters in $\langle W, R, D\rangle$.

Let $\mathfrak{F}(D) = \langle W, R, D\rangle$ be a predicate Kripke frame and let w be a world in it. A function α is called *interpretation of individual variables* in a world $w \in W$ if $\alpha(x_i) \in D(w)$, for every individual variable x_i.

Note that if w' is accessible from w and α is an interpretation of individual variables in w then α is an interpretation of individual variables in w', too, because in this case we have $D(w) \subseteq D(w')$.

For any individual variable x_i, we define the binary relation $\stackrel{x_i}{=}$ between interpretations. For interpretations α and β we put

$$\alpha \stackrel{x_i}{=} \beta \quad \leftrightharpoons \quad \alpha(x_k) = \beta(x_k), \text{ for any } k \in \mathbb{N} \text{ such that } k \neq i.$$

Let $\mathfrak{F} = \langle W, R\rangle$ be a serial Kripke frame, $\mathfrak{M} = \langle W, R, D, I\rangle$ be a Kripke model on $\mathfrak{F}$. We define the truth relation 'a formula φ is true at a world $w \in W$ in a model $\mathfrak{M}$ under an interpretation α of individual variables in w' inductively (by constructing of φ):

$(\mathfrak{M}, w) \not\models^\alpha \bot$;

$(\mathfrak{M}, w) \models^\alpha P_i^m(\overline{x}) \quad \leftrightharpoons \quad \alpha(\overline{x}) \in I(w, P_i^m)$ where $\overline{x} = (x_{k_1}, \ldots, x_{k_m})$, $\alpha(\overline{x}) = \langle \alpha(x_{k_1}), \ldots, \alpha(x_{k_m})\rangle$;

$(\mathfrak{M}, w) \models^\alpha \varphi_1 \wedge \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \models^\alpha \varphi_1$ and $(\mathfrak{M}, w) \models^\alpha \varphi_2$;

$(\mathfrak{M}, w) \models^\alpha \varphi_1 \vee \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \models^\alpha \varphi_1$ or $(\mathfrak{M}, w) \models^\alpha \varphi_2$;

$(\mathfrak{M}, w) \models^\alpha \varphi_1 \to \varphi_2 \quad \leftrightharpoons \quad (\mathfrak{M}, w) \not\models^\alpha \varphi_1$ or $(\mathfrak{M}, w) \models^\alpha \varphi_2$;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{AX}\varphi_1 \quad \leftrightharpoons$ for any path π starting in w the relation $(\mathfrak{M}, \pi_1) \models^\alpha \varphi_1$ is true;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{EX}\varphi_1 \quad \leftrightharpoons$ there is a path π starting in w such that $(\mathfrak{M}, \pi_1) \models^\alpha \varphi_1$;

$(\mathfrak{M}, w) \models^\alpha \boldsymbol{AF}\varphi_1 \quad \leftrightharpoons$ for any path π starting in w there is some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^\alpha \varphi_1$;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{EF}\varphi_1 \quad \risingdotseq \quad$ there are a path π starting in w and $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_1$;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{AG}\varphi_1 \quad \risingdotseq \quad$ for any path π starting in w and for any $k \in \mathbb{N}$ the relation $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \boldsymbol{EG}\varphi_1 \quad \risingdotseq \quad$ there is a path π starting in w such that for any $k \in \mathbb{N}$ the relation $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \varphi_1 \boldsymbol{AU} \varphi_2 \quad \risingdotseq \quad$ for any path π starting in w there is some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_2$ and, for any $j \in \mathbb{N}$, such that $j < k$ the relation $(\mathfrak{M}, \pi_j) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \varphi_1 \boldsymbol{EU} \varphi_2 \quad \risingdotseq \quad$ for some path π starting in w and some $k \in \mathbb{N}$ such that $(\mathfrak{M}, \pi_k) \models^{\alpha} \varphi_2$ and, for any $j \in \mathbb{N}$, such that $j < k$ the relation $(\mathfrak{M}, \pi_j) \models^{\alpha} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \forall x_i \varphi_1 \quad \risingdotseq \quad$ for any interpretation β such that $\beta \stackrel{x_i}{=} \alpha$ and $\beta(x_i) \in D(w)$ the relation $(\mathfrak{M}, w) \models^{\beta} \varphi_1$ is true;

$(\mathfrak{M}, w) \models^{\alpha} \exists x_i \varphi_1 \quad \risingdotseq \quad$ there is an interpretation β such that $\beta \stackrel{x_i}{=} \alpha$, $\beta(x_i) \in D(w)$, and $(\mathfrak{M}, w) \models^{\beta} \varphi_1$.

The relations 'φ is true at w in $\mathfrak{M}$', 'φ is true in $\mathfrak{M}$', 'φ is true in $\mathfrak{F}(D)$', and 'φ is true in $\mathfrak{F}$' are defined as follows:

$(\mathfrak{M}, w) \models \varphi \quad \risingdotseq \quad (\mathfrak{M}, w) \models^{\alpha} \varphi$, for any interpretation α of individual variables in w;

$\mathfrak{M} \models \varphi \quad \risingdotseq \quad (\mathfrak{M}, w) \models \varphi$, for any $w \in W$;

$\mathfrak{F}(D) \models \varphi \quad \leftrightharpoons \quad \mathfrak{M} \models \varphi$, for any model $\mathfrak{M}$ based on $\mathfrak{F}(D)$;

$\mathfrak{F} \models \varphi \quad \leftrightharpoons \quad \mathfrak{M} \models \varphi$, for any model $\mathfrak{M}$ based on $\mathfrak{F}$.

Note that any world w in a model $\mathfrak{M} = \langle W, R, D, I\rangle$ may be understood as a usual model for the classical first-order language. Indeed, as such model one may take the model $\mathfrak{M}_w = \langle D_w, I_w\rangle$ where $D_w = D(w)$ and $I_w(P_i^m) = I(w, P_i^m)$, for every predicate letter P_i^m.

We define the logic **QCTL** as the set of all temporal formulas that are true in any serial Kripke frame.

The logic **CTL** is a propositional fragment of **QCTL**.

Let also **QCL** denote the classical first-order logic in the modal-free fragment of the language for **QCTL** and let $\mathbf{QCL}_{fin}$ denote the logic of all finite models, i. e., the set of classical first-order formulas that are true in any model with finite domain.

3 Some facts

In this section we just recall some 'algorithmic' definitions and facts; so, the reader may omit this section.

Let U be some universal set (for example, a set of all formulas in some language). A set X is called *decidable* if there is an effective procedure (algorithm) A such that, for any $x \in U$,

$$A(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X; \end{cases}$$

otherwise X is called *undecidable*.

A set X is called *recursively enumerable* if $X = \varnothing$ or there is an effective procedure (algorithm) A such that $A(n)$ is defined for every $n \in \mathbb{N}$ and $X = \{A(n) : n \in \mathbb{N}\}$, in other words, if there is an effective enumeration for elements of X.

Note that if a logic is recursively (in particular, finitely) axiomatizable then it is recursively enumerable because in this case it is possible to enumerate effectively all derivations, and hence, all derivable formulas.

For a logic L, let $\overline{L}$ denote the complement of L in the set of all formulas in the language of L.

Below we will use the following facts, see [1, 11]:

- the logic $\mathbf{QCL}$ is undecidable; $\mathbf{QCL}$ is recursively enumerable and $\overline{\mathbf{QCL}}$ is not;
- the logic $\mathbf{QCL}_{fin}$ is undecidable; $\overline{\mathbf{QCL}}_{fin}$ is recursively enumerable and $\mathbf{QCL}_{fin}$ is not.

A set X is called *recursively reducible* to a set Y if there is an effective procedure (algorithm) A such that

$$x \in X \iff A(x) \in Y,$$

for any x (in the appropriate universal set U).

Let X be recursively reducible to Y. It is not hard to see that

- if Y is decidable then X is decidable;
- if Y is recursively enumerable then X is recursively enumerable.

4 Main aim

Let $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EG}, \boldsymbol{AF}, \boldsymbol{EF}, \boldsymbol{AU}, \boldsymbol{EU}\}$ and let X be a set of formulas (in some language). We use the denotation $X \upharpoonright M$ for the set of formulas in X those modalities belong to the set M^* (i. e., constructed only from modalities of M).

Our main aim is to describe some (algorithmic, semantical, deductive) properties of the logic $\mathbf{QCTL} \upharpoonright M$.

Of course, it is expected that properties of $\mathbf{QCTL} \upharpoonright M$ depend on M. Indeed, it is not hard to see that $\mathbf{QCTL} \upharpoonright \varnothing = \mathbf{QCL}$ and hence $\mathbf{QCTL} \upharpoonright \varnothing$ is finitely axiomatizable. But it follows from [12] that $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AX}, \boldsymbol{AG}\}$ is not recursively enumerable, wherefore it is not finitely (and even recursively) axiomatizable.

Below we prove the following statement.

Theorem 1. *Let M be a set of modalities including only some of $\boldsymbol{AX}$, $\boldsymbol{EX}$, $\boldsymbol{AG}$, $\boldsymbol{EG}$, $\boldsymbol{AF}$, $\boldsymbol{EF}$, $\boldsymbol{AU}$, $\boldsymbol{EU}$. Then the following equivalence holds:*

$$\begin{array}{c}\mathbf{QCTL} \upharpoonright M \text{ is recursively enumerable} \\ \Longleftrightarrow \\ M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\} \text{ or } M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}.\end{array}$$

Then, using Theorem 1 (and its proof) we will be able to prove some statements about algorithmic, semantical, and deductive properties of some first-order extensions of **CTL**.

5 Recursively enumerable fragments of QCTL

In this section we prove the following part of Theorem 1: if $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ or $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then $\mathbf{QCTL} \upharpoonright M$ is recursively enumerable.

To do this let us observe that

(a) $\mathbf{QCTL} \upharpoonright \varnothing = \mathbf{QCL}$;

(b) for any formula φ,

$$\begin{array}{rcl}\boldsymbol{AX}\varphi \leftrightarrow \neg\boldsymbol{EX}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{EX}\varphi \leftrightarrow \neg\boldsymbol{AX}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{AG}\varphi \leftrightarrow \neg\boldsymbol{EF}\neg\varphi & \in & \mathbf{QCTL}, \\ \boldsymbol{EF}\varphi \leftrightarrow \neg\boldsymbol{AG}\neg\varphi & \in & \mathbf{QCTL}.\end{array}$$

Because of (a), the logic $\mathbf{QCTL} \upharpoonright \varnothing$ is recursively enumerable and even finitely axiomatizable. Then, because of (b), it is sufficient to prove that $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AX}\}$ and $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AG}\}$ are recursively enumerable.

To prove the last statement we use the first-order modal logics **QD** and **QS4**.

The language of **QD** and **QS4** contains the same symbols as the language of **QCL** and the modality symbol $\Box$. Also we have the following extra rule for formula constructing: if φ is a formula then $\Box\varphi$ is a formula. We call formulas in this language *modal formulas*. The notions of Kripke frame and Kripke model are the same as they are defined in Section 2.

Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame, $\mathfrak{M} = \langle W, R, D, I\rangle$ be a Kripke model on $\mathfrak{F}$. Now we define the relation 'a modal formula φ is true at a world $w \in W$ in a model $\mathfrak{M}$ under an interpretation α of individual variables in w'. To differ the truth relation for modal formulas and the truth relation for temporal formulas we use the sign '$|\models$' for the first of them. This relation is defined inductively (by constructing of φ) in the same way as for temporal formulas, see page 72. The cases for the atomic formulas, for the connectives $\wedge$, $\vee$, $\to$, and for the quantifiers $\forall x_i$ and $\exists x_i$ are the same (the reader just must replace '$\models$' with '$|\models$'). As for $\Box$, the definition looks as follows:

$$(\mathfrak{M}, w) \mid\models^{\alpha} \Box\varphi_1 \quad \leftrightharpoons \quad \text{for any } w' \in W \text{ such that } wRw' \text{ the relation } (\mathfrak{M}, w') \mid\models^{\alpha} \varphi_1 \text{ is true.}$$

The relations $(\mathfrak{M}, w) \mid\models \varphi$, $\mathfrak{M} \mid\models \varphi$, $\mathfrak{F}(D) \mid\models \varphi$, and $\mathfrak{F} \mid\models \varphi$ are defined as on page 73.

Now we define the logic **QD** as the set of all modal formulas that are true in any serial frame and the logic **QS4** as the set of all modal formulas that are true in any reflexive and transitive frame.

It is known that **QD** and **QS4** are finitely axiomatizable and, in particular, recursively enumerable, see [3]. We are going to show that **QCTL** $\restriction \{\boldsymbol{AX}\}$ is recursively reducible to **QD** and **QCTL** $\restriction \{\boldsymbol{AG}\}$ is recursively reducible to **QS4**.

Let us define translations T_1 and T_2. Suppose a temporal formula φ does not contain modalities different from $\boldsymbol{AX}$ and its iterations; then define $T_1(\varphi)$ to be a modal formula obtained from φ by replacing every occurrence of $\boldsymbol{AX}$ with $\Box$. Let φ does not contain modalities different from $\boldsymbol{AG}$ and its iterations; then define $T_2(\varphi)$ to be a modal formula obtained from φ by replacing every occurrence of $\boldsymbol{AG}$ with $\Box$.

OBSERVATION 1. *For any temporal formula φ without modalities different from $\boldsymbol{AX}$ and its iterations, the following equivalence holds:*

$$\varphi \in \mathbf{QCTL} \iff T_1(\varphi) \in \mathbf{QD},$$

i. e., T_1 recursively reduces **QCTL** $\restriction \{\boldsymbol{AX}\}$ *to* **QD**.

Proof. Let φ be a formula without modalities different from $\boldsymbol{AX}$ and its iterations.

Let $\varphi \notin \mathbf{QCTL}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I \rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$.

In this case, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w, the following equivalence holds:

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}, w) \,|\!\!\models^{\alpha} T_1(\psi).$$

The proof proceeds by induction on constructing of ψ and we left the details to the reader.

As a result we obtain that $(\mathfrak{M}, w_0) \,|\!\!\not\models^{\alpha_0} T_1(\varphi)$ and hence, $T_1(\varphi) \notin \mathbf{QD}$.

Let $T_1(\varphi) \notin \mathbf{QD}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I \rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \,|\!\!\not\models^{\alpha_0} T_1(\varphi)$. With the same argumentation we obtain that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$, therefore, $\varphi \notin \mathbf{QCTL}$. □

Observation 2. *For any temporal formula φ without modalities different from $\boldsymbol{AG}$ and its iterations, the following equivalence holds:*

$$\varphi \in \mathbf{QCTL} \iff T_2(\varphi) \in \mathbf{QS4},$$

i. e., T_2 recursively reduces $\mathbf{QCTL} \restriction \{\boldsymbol{AG}\}$ to $\mathbf{QS4}$.

Proof. Let φ be a formula without modalities different from $\boldsymbol{AG}$ and its iterations.

Let $\varphi \notin \mathbf{QCTL}$. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I \rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$.

Let R^* be reflexive and transitive closure of R and let $\mathfrak{M}^* = \langle W, R^*, D, I \rangle$. Then, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w, the following equivalence holds:

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}^*, w) \,|\!\!\models^{\alpha} T_2(\psi).$$

The proof proceeds by induction on constructing of ψ and we again left the details to the reader.

In particular, we have $(\mathfrak{M}^*, w_0) \mid\not\models^{\alpha_0} T_2(\varphi)$. Because $\mathfrak{M}^*$ is a model for **QS4**, we obtain $T_2(\varphi) \notin \mathbf{QS4}$.

Let $T_2(\varphi) \notin \mathbf{QS4}$. Then there is a reflexive and transitive model $\mathfrak{M} = \langle W, R, D, I\rangle$, a world $w_0 \in W$, and an interpretation α_0 of individual variables in w_0 such that $(\mathfrak{M}, w_0) \mid\not\models^{\alpha_0} T_1(\varphi)$. Because $\mathfrak{M}$ is reflexive, it is serial. Then, for any formula ψ, any $w \in W$, and any interpretation α of individual variables in w,

$$(\mathfrak{M}, w) \models^{\alpha} \psi \iff (\mathfrak{M}, w) \mid\models^{\alpha} T_2(\psi),$$

and hence, $(\mathfrak{M}, w_0) \not\models^{\alpha_0} \varphi$. Thus, $\varphi \notin \mathbf{QCTL}$. □

Because the logics **QD** and **QS4** are recursively enumerable, from Observations 1 and 2 it follows that $\mathbf{QCTL} \restriction \{\boldsymbol{AX}\}$ and $\mathbf{QCTL} \restriction \{\boldsymbol{AG}\}$ are recursively enumerable. In fact, even more strong result holds.

PROPOSITION 1. *The temporal logics* $\mathbf{QCTL} \restriction \{\boldsymbol{AX}\}$ *and* $\mathbf{QCTL} \restriction \{\boldsymbol{AG}\}$ *are finitely axiomatizable.*

PROOF. It is sufficient to observe that the translation T_1 is an isomorphism between $\mathbf{QCTL} \restriction \{\boldsymbol{AX}\}$ and **QD**, the translation T_2 is an isomorphism between $\mathbf{QCTL} \restriction \{\boldsymbol{AG}\}$ and **QS4**. □

6 Main technical construction

Now we start to prove that if $M \not\subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ and $M \not\subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then $\mathbf{QCTL} \restriction M$ is not recursively enumerable.

Here we present some technical constructions and statements; then we apply them to achieve the main aim.

Let us fix three binary letters and one unary letter; to make it easier to understand (and with according of their intending meaning) we denote them $\approx$, $\prec$, $\prec_1$, and L. To explain the intending meaning of these predicate letters, let us imagine that we try to order equivalence classes of some set; then

$x \approx y$	means	‘x and y are equivalent’;
$x \prec y$	means	‘x is less than y’;
$x \prec_1 y$	means	‘y is the next element after x’;
$L(x)$	means	‘x is a label (for a current state)’.

With help of 'labels' and modalities we show how to describe the condition that equivalence classes are ordered as positive integers by the relation 'less than'.

Let us define some formulas. The formula A_1 claims $\approx$ to be an equivalence relation:

$$\begin{aligned} A_1 &= \forall x(x \approx x) \wedge \forall x \forall y(x \approx y \to y \approx x) \wedge \\ &\qquad\qquad \forall x \forall y \forall z(x \approx y \wedge y \approx z \to x \approx z). \end{aligned}$$

The formula A_2 claims $\approx$ to be a congruence relative to $\prec$:

$$A_2 = \forall x \forall y \forall u \forall v(x \approx u \wedge y \approx v \to (x \prec y \to u \prec v)).$$

The formula A_3 claims $\prec$ to be a strict linear order (on equivalence classes):

$$\begin{aligned} A_3 &= \forall x\, \neg(x \prec x) \wedge \\ &\qquad \forall x \forall y \forall z(x \prec y \wedge y \prec z \to x \prec z) \wedge \\ &\qquad\qquad \forall x \forall y(x \prec y \vee x \approx y \vee y \prec x). \end{aligned}$$

The formula A_4 defines $\prec_1$ as a successor relation with respect to $\prec$:

$$A_4 = \forall x \forall y((x \prec_1 y) \leftrightarrow (x \prec y \wedge \neg \exists z(x \prec z \wedge z \prec y))).$$

The formula A_5 means that any element has a successor:

$$A_5 = \forall x \exists y(x \prec_1 y).$$

Let $x \not\approx y$ be the abbreviation for $\neg(x \approx y)$. The formulas A_6 and A_7 claim heredity for $\prec_1$ and $\not\approx$, correspondingly:

$$\begin{aligned} A_6 &= \forall x \forall y(x \prec_1 y \to \boldsymbol{AG}(x \prec_1 y)); \\ A_7 &= \forall x \forall y(x \not\approx y \to \boldsymbol{AG}(x \not\approx y)). \end{aligned}$$

The formula A_8 means for a world that it has a unique label (if it has a label at all):

$$A_8 = \forall x \forall y(L(x) \wedge L(y) \to x \approx y).$$

The formula A_9 means that any next world has the next label:

$$A_9 = \forall x \forall y(L(x) \wedge x \prec_1 y \to \boldsymbol{AX} L(y)).$$

Let

$$A \quad = \quad \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_9).$$

Let also

$$B \quad = \quad \exists x(L(x) \wedge \neg\exists y(y \prec x)).$$

The formula B means that there is a least element ('zero') and it is a label (in a world where B is true). Finally, let

$$C \quad = \quad \forall x \boldsymbol{EF} L(x).$$

The formula C means that any element (of some current world) labels some future world.

As we will see below, the formula $A \wedge B \wedge C$ provides us with a condition that is sufficient to prove that equivalence classes are ordered with $\prec$ as positive integers with the relation 'less than'. But before this we show that the formula $A \wedge B \wedge C$ is **QCTL**-satisfiable, i. e., $\neg(A \wedge B \wedge C) \notin$ **QCTL**.

Let $\mathfrak{M}_0 = \langle W_0, R_0, D_0, I_0 \rangle$ where

$$\begin{array}{lcl}
W_0 & = & \{w_i : i \in \mathbb{N}\}; \\
w_i R_0 w_j & \leftrightharpoons & j = i + 1; \\
D_0(w_i) & = & \mathbb{N}; \\
I_0(w_i, \approx) & = & \{\langle m, m \rangle : m \in \mathbb{N}\}; \\
I_0(w_i, \prec) & = & \{\langle m, k \rangle : m, k \in \mathbb{N} \text{ and } m < k\}; \\
I_0(w_i, \prec_1) & = & \{\langle m, m+1 \rangle : m \in \mathbb{N}\}; \\
I_0(w_i, L) & = & \{\langle i \rangle\}.
\end{array}$$

Lemma 1. *It is true that* $(\mathfrak{M}_0, w_0) \models A \wedge B \wedge C$*.*

Proof is straightforward and left to the reader. □

Proposition 2. *The formula* $A \wedge B \wedge C$ *is* **QCTL**-*satisfiable.*

Proof. It is sufficient to observe that the accessibility relation in the model $\mathfrak{M}_0$ is serial and then use Lemma 1. □

Now let us turn to the key technical lemmas of the article.

Let $\mathfrak{M} = \langle W, R, D, I\rangle$ be a serial model and w^* be a world in W such that $(\mathfrak{M}, w^*) \models A \wedge B \wedge C$.

For simplicity let us use the following abbreviations, for any $w \in W$:

$$\approx^w = I(w, \approx), \ \ \prec^w = I(w, \prec), \ \ \prec_1^w = I(w, \prec_1), \ \ L^w = I(w, L).$$

Let also, for any $w, w' \in W$ and any $k \in \mathbb{N}$,

$$\begin{array}{lcl} wR^0w' & \leftrightharpoons & w = w'; \\ wR^{k+1}w' & \leftrightharpoons & wR^k u \text{ and } uRw', \text{ for some } u \in W; \\ wR^*w' & \leftrightharpoons & wR^m w', \text{ for some } m \in \mathbb{N}. \end{array}$$

Observe that $(\mathfrak{M}, w) \models A_1 \wedge \ldots \wedge A_9$, for any $w \in W$ such that w^*R^*w; this is so because $(\mathfrak{M}, w^*) \models \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_9)$.

Lemma 2. *Let $w \in W$ and w^*R^*w. Then the relation $\approx^w$ is a congruence with respect to the relation $\prec^w$.*

Proof immediately follows from $(\mathfrak{M}, w) \models A_1 \wedge A_2$. □

Because of Lemma 2 we may define congruence classes: for any $w \in W$ such that w^*R^*w and any $a \in D(w)$ we put

$$[a]^w \ = \ \{b \in D(w) : b \approx^w a\}.$$

Note that the relations $\prec^w$ and $\prec_1^w$ on $D(w)$ generate the relations $\prec^w$ and $\prec_1^w$ on equivalence classes defined in the following way:

$$\begin{array}{lcl} [a]^w \prec^w [b]^w & \leftrightharpoons & a \prec^w b; \\ [a]^w \prec_1^w [b]^w & \leftrightharpoons & a \prec_1^w b, \end{array}$$

for any $w \in W$ such that w^*R^*w and for any $a, b \in D(w)$. Let also

$$[a]^w \preccurlyeq^w [b]^w \ \leftrightharpoons \ [a]^w \prec^w [b]^w \text{ or } [a]^w = [b]^w.$$

Let, for any $w \in W$ such that w^*R^*w,

$$D^w \ = \ \{[a]^w : a \in D(w)\}.$$

For the rest of this section our aim is to prove that $\langle D^{w^*}, \prec^{w^*}\rangle$ is isomorphic to $\langle \mathbb{N}, <\rangle$.

LEMMA 3. *Let $w \in W$ and $w^* R^* w$. Then the relation $\prec^w$ is a strict linear order on D^w and $\prec_1^w$ is the successor relation on D^w with respect to $\prec^w$.*

PROOF immediately follows from $(\mathfrak{M}, w) \models A_3 \wedge A_4$. □

Since $(\mathfrak{M}, w^*) \models B$, there is $a_0 \in D(w^*)$ such that $L^{w^*}(a_0)$ and $[a_0]^{w^*} \preccurlyeq^{w^*} [a]^{w^*}$, for any $a \in D(w^*)$. Then, due to $(\mathfrak{M}, w^*) \models A_5$, there are $a_1, a_2, a_3, \ldots \in D(w^*)$ such that

$$a_0 \prec_1^{w^*} a_1 \prec_1^{w^*} a_2 \prec_1^{w^*} a_3 \prec_1^{w^*} \ldots;$$

note by the way that equivalence classes generated by elements $a_0, a_1, a_2, a_3, \ldots$ in $D(w^*)$ are pairwise different.

LEMMA 4. *Let $w \in W$ and $w^* R^* w$. Then*

$$\begin{aligned} a \prec_1^{w^*} b &\implies a \prec_1^{w} b; \\ a \not\approx^{w^*} b &\implies a \not\approx^{w} b, \end{aligned}$$

for any $a, b \in D(w^*)$.

PROOF immediately follows from $(\mathfrak{M}, w^*) \models A_6 \wedge A_7$. □

LEMMA 5. *Let $w \in W$, $k \in \mathbb{N}$, and $w^* R^k w$. Then $L^w(a_k)$ is true.*

PROOF proceeds by induction on k.

Let $k = 0$. Then we must prove that $L^{w^*}(a_0)$ is true; but we have $L^{w^*}(a_0)$ to be true because of choosing of a_0.

Let the statement be true for k; we prove it for $k+1$. Let $w^* R^{k+1} w$. Then there is $u \in W$ such that $w^* R^k u$ and uRw. By induction hypothesis, $L^u(a_k)$ holds. By Lemma 4, we have $a_k \prec_1^u a_{k+1}$. Then, from $(\mathfrak{M}, u) \models A_9$ and uRw we obtain $L^w(a_{k+1})$. □

LEMMA 6. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_k$, for some $k \in \mathbb{N}$.*

PROOF. Because of $(\mathfrak{M}, w^*) \models C$, there is $w \in W$ such that $w^* R^k w$, for some $k \in \mathbb{N}$, and $L^w(b)$ is true. By Lemma 5, $L^w(a_k)$ is true. Then, from $(\mathfrak{M}, w) \models A_8$ we obtain $b \approx^w a_k$ and, by Lemma 4, we obtain $b \approx^{w^*} a_k$. □

As a corollary we obtain the next statement.

PROPOSITION 3. *The structures* $\langle D^{w^*}, \prec^{w^*} \rangle$ *and* $\langle \mathbb{N}, < \rangle$ *are isomorphic.*

PROOF. For any $k \in \mathbb{N}$, let $f(k) = [a_k]^{w^*}$. We show that f is the required isomorphism. Clearly, $k < m$ implies $[a_k]^{w^*} \prec^{w^*} [a_m]^{w^*}$. So, we must prove that f is injective and surjective.

Injectivity of f. Let $k \neq m$; without a loss of generality we may assume that $k < m$. But then $[a_k]^{w^*} \prec^{w^*} [a_m]^{w^*}$ and hence $f(k) \neq f(m)$.

Surjectivity of f. Let $b \in D(w^*)$. Then, by Lemma 6, we have $b \approx^{w^*} a_k$, for some $k \in \mathbb{N}$; this means that $[b]^{w^*} = [a_k]^{w^*} = f(k)$. □

7 Embedding of $\mathbf{QCL}_{fin}$ into QCTL

Let φ be some closed classical first-order formula and let y be some individual variable not occurring in φ. Let also $x \preccurlyeq y$ be an abbreviation for the formula $(x \prec y \vee x \approx y)$. We define the translation T_y:

$$\begin{array}{lcl}
T_y(\bot) & = & \bot; \\
T_y(P_i^m(x_{k_1}, \ldots, x_{k_m})) & = & P_i^m(x_{k_1}, \ldots, x_{k_m}); \\
T_y(\psi' \wedge \psi'') & = & T_y(\psi') \wedge T_y(\psi''); \\
T_y(\psi' \vee \psi'') & = & T_y(\psi') \vee T_y(\psi''); \\
T_y(\psi' \to \psi'') & = & T_y(\psi') \to T_y(\psi''); \\
T_y(\forall x \psi') & = & \forall x (x \preccurlyeq y \to T_y(\psi')); \\
T_y(\exists x \psi') & = & \exists x (x \preccurlyeq y \wedge T_y(\psi'))
\end{array}$$

where ψ' and ψ'' are subformulas of the formula φ. Then we put $T(\varphi) = \forall y \, T_y(\varphi)$.

To explain the intending meaning of $T_y(\varphi)$ and $T(\varphi)$ let us imagine that we interpret individual variables as positive integers and $\preccurlyeq$ as the relation $\leqslant$ on $\mathbb{N}$. Then $T_y(\varphi)$ means 'φ is true in any model with elements $0, \dots, y$' and $T(\varphi)$ means 'φ is true in any finite model'.

Let $P_{i_1}^{m_1}, \dots, P_{i_k}^{m_k}$ be the list of all predicate letters occurring in φ. We define the formula $Congr(\varphi)$ as a conjunction of formulas in the following form:

$$\forall x_1 \dots \forall x_{m_j} \forall y_1 \dots \forall y_{m_j} \big(\bigwedge_{i=1}^{m_j} (x_i \approx y_i) \to \\ \to (P_{i_j}^{m_j}(x_1, \dots, x_{m_j}) \to P_{i_j}^{m_j}(y_1, \dots, y_{m_j}))\big)$$

where $j \in \{1, \dots, k\}$. Let

$$Emb(\varphi) \quad = \quad A \wedge B \wedge C \wedge Congr(\varphi) \to T(\varphi).$$

LEMMA 7. $\varphi \in \mathbf{QCL}_{fin} \iff Emb(\varphi) \in \mathbf{QCTL}$.

PROOF. Suppose $\varphi \notin \mathbf{QCL}_{fin}$. Then there is a classical model $\mathfrak{S} = \langle S, J \rangle$ where S is a finite set, J is an interpretation of predicate letters in S, such that $\mathfrak{S} \not\models \varphi$. Without a loss of generality we may assume that $S = \{0, \dots, n\}$, for some $n \in \mathbb{N}$.

Let $\mathfrak{M}_0$ be Kripke model defined on page 81. We extend I_0 on predicate letters occurring in φ; let

$$I_0(w_0, P_{i_j}^{m_j}) \quad = \quad \{\langle k_1, \dots, k_{m_j} \rangle : \langle k_1, \dots, k_{m_j} \rangle \in J(P_{i_j}^{m_j})\},$$

for any $j \in \{1, \dots, k\}$, i.e., $I_0(w_0, P_{i_j}^{m_j}) = J(P_{i_j}^{m_j})$. Let also $I_0(w_i, P_{i_j}^{m_j}) = \varnothing$, for any $i \in \mathbb{N}^+$.

Then we claim $(\mathfrak{M}_0, w_0) \not\models Emb(\varphi)$. Since the letter $\approx$ is interpreted with the identity relation on $D(w_0)$, it is clear that $(\mathfrak{M}_0, w_0) \models Congr(\varphi)$. By Lemma 1, we have $(\mathfrak{M}_0, w_0) \models A \wedge B \wedge C$. Hence we just must prove that $(\mathfrak{M}_0, w_0) \not\models \forall y\, T_y(\varphi)$. It is sufficient to interpret y as n. More exactly, the following condition holds: for any subformula ψ of the formula φ, for any interpretation α such that $\alpha(x) \in \{0, \dots, n\}$, for any individual variable x, and $\alpha(y) = n$,

$$\mathfrak{S} \models^\alpha \psi \iff (\mathfrak{M}_0, w_0) \models^\alpha T_y(\psi).$$

We left the proof of the condition to the reader; it proceeds by induction on constructing of ψ. Thus, $(\mathfrak{M}_0, w_0) \not\models^\alpha T_y(\varphi)$, for any such interpretation α. Therefore $(\mathfrak{M}_0, w_0) \not\models \forall y\, T_y(\varphi)$ and hence $(\mathfrak{M}_0, w_0) \not\models Emb(\varphi)$.

Since the model $\mathfrak{M}_0$ is serial, we have $Emb(\varphi) \notin$ **QCTL**.

Now suppose $Emb(\varphi) \notin$ **QCTL**. Then there is a serial model $\mathfrak{M} = \langle W, R, D, I\rangle$ such that $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$, for some $w^* \in W$. From $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$ we obtain $(\mathfrak{M}, w^*) \models A \wedge B \wedge C$ and hence we may use results of Section 6. Let

$$\approx^{w^*} = I(w^*, \approx), \quad \prec^{w^*} = I(w^*, \prec).$$

Let also

$$\begin{array}{rcl} [a] & = & \{b \in D(w^*) : b \approx^{w^*} a\}; \\ D^{w^*} & = & \{[a] : a \in D(w^*)\}; \\ [a] \prec^{w^*} [b] & \leftrightharpoons & a \prec^{w^*} b. \end{array}$$

Then, by Proposition 3, the structures $\langle D^{w^*}, \prec^{w^*}\rangle$ and $\langle \mathbb{N}, <\rangle$ are isomorphic. Let $g : D^{w^*} \to \mathbb{N}$ be an isomorphism between the structures. Let us define an interpretation J for predicate letters occurring in $T(\varphi)$. Let P be a predicate letter occurring in $T(\varphi)$, i.e., P is one of the letters $P^{m_1}_{i_1}, \ldots, P^{m_k}_{i_k}$, $\approx$, and $\prec$, and let m be the arity of P. We put

$$J(P) = \{\langle g([b_1]), \ldots, g([b_m])\rangle : \langle b_1, \ldots, b_m\rangle \in I(w^*, P)\}.$$

From $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$ it follows that $(\mathfrak{M}, w^*) \models Congr(\varphi)$ and hence the relation $\approx^{w^*}$ is a congruence relative to $I(w^*, P^{m_j}_{i_j})$, for any $j \in \{1, \ldots, k\}$. This means that J is well defined. Note that, in particular, $J(\approx)$ is the identity relation on $\mathbb{N}$ and $J(\prec)$ is the relation $<$ on $\mathbb{N}$.

Let $\mathfrak{S} = \langle \mathbb{N}, J\rangle$.

We claim $\mathfrak{S} \not\models T(\varphi)$. Let α be an interpretation of individual variables in $D(w^*)$ and β be an interpretation of individual variables in $\mathbb{N}$; we call α and β *agreed* interpretations if $\beta(x) = g([\alpha(x)])$, for any variable x. Then

$$(\mathfrak{M}_0, w_0) \models^{\alpha} \chi \iff \mathfrak{S} \models^{\beta} \chi,$$

for any agreed interpretations α and β and for any classical formula χ containing no predicate letters different from $P^{m_1}_{i_1}, \ldots, P^{m_k}_{i_k}$, $\approx$, and $\prec$; the details are left to the reader. Since $(\mathfrak{M}, w^*) \not\models Emb(\varphi)$, we have $(\mathfrak{M}, w^*) \not\models T(\varphi)$, and hence, $\mathfrak{S} \not\models T(\varphi)$.

It follows from $\mathfrak{S} \not\models T(\varphi)$ that $\mathfrak{S} \not\models^{\alpha} T_y(\varphi)$, for some interpretation α. Let $n = \alpha(y)$ and $\mathfrak{S}' = \langle\{0, \dots, n\}, J'\rangle$ where J' is a restriction of J on $\{0, \dots, n\}$. Then, for any subformula ψ of the formula φ, for any interpretation β such that $\beta(x) \in \{0, \dots, n\}$, for any variable x, and $\beta(y) = n$,

$$\mathfrak{S}' \models^{\beta} \psi \iff \mathfrak{S} \models^{\beta} T_y(\psi).$$

The details of the proof are left to the reader.

Since $\mathfrak{S} \not\models^{\alpha} T_y(\varphi)$ we obtain $\mathfrak{S}' \not\models \varphi$. The model $\mathfrak{S}'$ is finite, therefore $\varphi \notin \mathbf{QCL}_{fin}$. □

COROLLARY 1. *The logic* **QCTL** *is not recursively enumerable.*

COROLLARY 2. *The logic* $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AX}, \boldsymbol{AG}\}$ *is not recursively enumerable.*

8 Modifications for other fragments

Now we have got a 'weak Theorem 1': just for the case $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EF}\}$. What about other modalities? In this section we show that the fragment of **QCTL** with any of the modalities $\boldsymbol{AF}$, $\boldsymbol{EG}$, $\boldsymbol{AU}$, and $\boldsymbol{EU}$ is not recursively enumerable. Since, for any formula φ,

$$\begin{array}{lcl} \boldsymbol{EG}\varphi \leftrightarrow \neg\boldsymbol{AF}\neg\varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{AF}\varphi \leftrightarrow \neg\boldsymbol{EG}\neg\varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{AF}\varphi \leftrightarrow \top\boldsymbol{AU}\varphi & \in & \mathbf{QCTL}, \end{array}$$

it is sufficient to prove that $\mathbf{QCTL} \upharpoonright \{\boldsymbol{EU}\}$ is not recursively enumerable and that at least one of $\mathbf{QCTL} \upharpoonright \{\boldsymbol{AF}\}$ and $\mathbf{QCTL} \upharpoonright \{\boldsymbol{EG}\}$ is not recursively enumerable. We consider the first and the second fragments.

To prove that these fragments are not recursively enumerable, it is sufficient to construct some embeddings of $\mathbf{QCL}_{fin}$ into the fragments. To construct such embeddings we modify the embedding Emb defined on page 85. Our modifications do not concern 'classical' part of $Emb(\varphi)$, i. e., we modify only the formulas A, B, and C defined in Section 6. The aim is to prove a statement like Lemma 7. Observe that the formulas A, B, and C are used in the proof of Lemma 7 only once: we need just Proposition 3 to be true. Therefore for our purposes it is sufficient to modify A, B, and C so that Proposition 3 remains to be true for resulting formulas. Finally, observe that the key condition in the proof of Proposition 3 is Lemma 6.

8.1 Fragments with $\boldsymbol{EU}$

Let $\boldsymbol{AR}$ be the dual modality for $\boldsymbol{EU}$; we define it as the following abbreviation:

$$(\varphi \boldsymbol{AR} \psi) \quad = \quad \neg(\neg\varphi \boldsymbol{EU} \neg\psi),$$

for any formulas φ and ψ. Observe that, for any φ,

$$\begin{array}{rcl} \boldsymbol{AG}\varphi \leftrightarrow \bot \boldsymbol{AR} \varphi & \in & \mathbf{QCTL}; \\ \boldsymbol{EF}\varphi \leftrightarrow \top \boldsymbol{EU} \varphi & \in & \mathbf{QCTL}, \end{array}$$

and therefore we may use $\boldsymbol{AG}$ and $\boldsymbol{EF}$ as corresponding abbreviations. Then, we may define all formulas $A_1, \ldots, A_8$, B, and C but not A_9 because A_9 contains the modality $\boldsymbol{AX}$. Instead of A_9 we take the formula A_9':

$$A_9' \quad = \quad \forall x \forall y \big(L(x) \wedge x \prec_1 y \to L(y) \boldsymbol{AR} (L(x) \vee L(y))\big).$$

Let $A' = \boldsymbol{AG}(A_1 \wedge \ldots \wedge A_8 \wedge A_9')$.

Proposition 4. *The formula $A' \wedge B \wedge C$ is $\mathbf{QCTL}$-satisfiable.*

Proof. It is sufficient to check that $(\mathfrak{M}_0, w_0) \models \boldsymbol{AG} A_9'$ and use Lemma 1. ☐

Let us use the notations introduced in Section 6 where $\mathfrak{M}$ is a model and w^* is a world in it such that $(\mathfrak{M}, w^*) \models A' \wedge B \wedge C$. It is clear that Lemmas 2, 3, and 4 are true. Lemma 5 is not true but we will prove a similar statement.

Lemma 8. *Let $w \in W$, $k \in \mathbb{N}$, and $w^* R^k w$. Then $L^w(a_m)$ is true, for some $m \in \{0, \ldots, k\}$.*

PROOF proceeds by induction on k. If $k = 0$ then with the same argumentation as in Lemma 5 we obtain $L^{w^*}(a_0)$.

Let the statement is true for k; we prove it for $k+1$. Let $w^* R^{k+1} w$. Then there is $u \in W$ such that $w^* R^k u$ and uRw. By induction hypothesis, there is $m \in \{0, \ldots, k\}$ such that $L^u(a_m)$ holds. By Lemma 4, we have $a_m \prec_1^u a_{m+1}$. Then, from $(\mathfrak{M}, u) \models A_9'$ and uRw we obtain $L^w(a_m)$ or $L^w(a_{m+1})$. $\square$

The next lemma is like Lemma 6.

LEMMA 9. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_m$, for some $m \in \mathbb{N}$.*

PROOF. Because of $(\mathfrak{M}, w^*) \models C$, there is $w \in W$ such that $w^* R^k w$, for some $k \in \mathbb{N}$, and $L^w(b)$ is true. By Lemma 8, $L^w(a_m)$ is true, for some $m \in \{0, \ldots, k\}$. Then, from $(\mathfrak{M}, w) \models A_8$ we obtain $b \approx^w a_m$ and, by Lemma 4, we obtain $b \approx^{w^*} a_k$. $\square$

From Lemma 9 we obtain a statement like Proposition 3.

PROPOSITION 5. *The structures $\langle D^{w^*}, \prec^{w^*} \rangle$ and $\langle \mathbb{N}, < \rangle$ are isomorphic.*

PROOF. Follow to the proof of Proposition 3. $\square$

Let φ be some closed classical first-order formula. We define $Emb'(\varphi)$:

$$Emb'(\varphi) \quad = \quad A' \wedge B \wedge C \wedge Congr(\varphi) \rightarrow T(\varphi).$$

LEMMA 10. $\varphi \in \mathbf{QCL}_{fin} \iff Emb'(\varphi) \in \mathbf{QCTL}$.

PROOF proceeds with the same argumentation as the proof for Lemma 7; the difference is in use Proposition 5 instead of Proposition 3. $\square$

COROLLARY 3. *The logic **QCTL** $\upharpoonright \{\boldsymbol{EU}\}$ is not recursively enumerable.*

8.2 Fragments with $\boldsymbol{AF}$

Observe that we may use the modality $\boldsymbol{EG}$ as an abbreviation because $\boldsymbol{EG}\varphi \leftrightarrow \neg\boldsymbol{AF}\neg\varphi \in \mathbf{QCTL}$, for any formula φ.

Let p be a propositional variable (i. e., some 0-ary predicate letter). We need it to define 'heredity' formulas:

$$\begin{array}{rcl} A_6'' & = & \forall x\forall y\big((x \prec_1 y \leftrightarrow \boldsymbol{AF}(x \prec_1 y \wedge p)) \wedge \\ & & \quad (x \prec_1 y \leftrightarrow \boldsymbol{AF}(x \prec_1 y \wedge \neg p))\big); \\ A_7'' & = & \forall x\forall y\big((x \not\approx y \leftrightarrow \boldsymbol{AF}(x \not\approx y \wedge p)) \wedge \\ & & \quad (x \not\approx y \leftrightarrow \boldsymbol{AF}(x \not\approx y \wedge \neg p))\big). \end{array}$$

Let us also define A_9'':

$$\begin{array}{rcl} A_9'' & = & \forall x\forall y\big(L(x) \wedge x \prec_1 y \to \boldsymbol{AF}L(y)\big) \wedge \\ & & \qquad \forall y\big(\boldsymbol{AF}L(y) \to \exists x(x \preccurlyeq y \wedge L(x))\big) \end{array}$$

where $x \preccurlyeq y$ is an abbreviation for $(x \prec y \vee x \approx y)$. Finally, let

$$\begin{array}{rcl} A'' & = & A_1 \wedge \ldots \wedge A_5 \wedge A_6'' \wedge A_7'' \wedge A_8 \wedge A_9''; \\ B'' & = & \exists x\big(L(x) \wedge \boldsymbol{EG}(A'' \wedge \neg\exists y(y \prec x))\big); \\ C'' & = & \forall x\boldsymbol{AF}L(x). \end{array}$$

Proposition 6. *The formula $B'' \wedge C''$ is* **QCTL***-satisfiable.*

Proof. Let $\mathfrak{M}_0$ be the model defined on page 6. We extend I_0 on p so that

$$(\mathfrak{M}_0, w_i) \models p \iff i = 2k, \text{ for some } k \in \mathbb{N},$$

i. e., we put p to be true exactly in 'even' worlds.

Then $(\mathfrak{M}_0, w_0) \models B'' \wedge C''$; corresponding check is left to the reader. $\square$

Let us use again the notations introduced in Section 6.

Let $\mathfrak{M} = \langle W, R, D, I\rangle$ be a model and w^* be a world in it such that $(\mathfrak{M}, w^*) \models B'' \wedge C''$.

Since $(\mathfrak{M}, w^*) \models B''$, there are $a_0 \in D(w^*)$ and a path π starting in w^* such that

- $L^{\pi_0}(a_0)$ is true;
- $b \prec^{\pi_k} a_0$ does not true, for any $k \in \mathbb{N}$ and for any $b \in D(\pi_k)$;
- $(\mathfrak{M}, \pi_k) \models A''$, for any $k \in \mathbb{N}$.

LEMMA 11. *The relation $\approx^{\pi_k}$ is a congruence with respect to the relation $\prec^{\pi_k}$, for any $k \in \mathbb{N}$.*

PROOF immediately follows from $(\mathfrak{M}, \pi_k) \models A_1 \wedge A_2$. □

Because of Lemma 11, we again may define congruence classes but just for worlds in the path π.

LEMMA 12. *The relation $\prec^{\pi_k}$ is a strict linear order on D^{π_k} and $\prec_1^{\pi_k}$ is the successor relation on D^{π_k} with respect to $\prec^{\pi_k}$, for any $k \in \mathbb{N}$.*

PROOF immediately follows from $(\mathfrak{M}, \pi_k) \models A_3 \wedge A_4$. □

From the condition $(\mathfrak{M}, \pi_0) \models A_5$ it follows that there are $a_1, a_2, a_3, \ldots \in D(\pi_0)$ such that

$$a_0 \prec^{\pi_0} a_1 \prec^{\pi_0} a_2 \prec^{\pi_0} a_3 \prec^{\pi_0} \ldots$$

and equivalence classes generated by $a_0, a_1, a_2, a_3, \ldots$ are pairwise different.

LEMMA 13. *For any $k \in \mathbb{N}$ and for any $a, b \in D(\pi_0)$,*

$$\begin{array}{rcl} a \prec_1^{\pi_0} b & \Longrightarrow & a \prec_1^{\pi_k} b; \\ a \not\approx^{\pi_0} b & \Longrightarrow & a \not\approx^{\pi_k} b. \end{array}$$

PROOF. Let $a \prec_1^{\pi_0} b$, for some $a, b \in D(\pi_0)$. We prove that $a \prec_1^{\pi_k} b$ by induction on k. If $k = 0$ then the statement is trivial.

Let $a \prec_1^{\pi_k} b$ and let α be an interpretation of variables in $D(\pi_k)$ such that $\alpha(x) = a$, $\alpha(y) = b$. There are two possible cases: $(\mathfrak{M}, \pi_k) \models^{\alpha} p$ and $(\mathfrak{M}, \pi_k) \not\models^{\alpha} p$.

Case $(\mathfrak{M}, \pi_k) \models^{\alpha} p$. Suppose $a \not\prec_1^{\pi_{k+1}} b$. Since $(\mathfrak{M}, \pi_{k+1}) \models A_6$, we have $(\mathfrak{M}, \pi_{k+1}) \not\models^{\alpha} \boldsymbol{AF}(x \prec y \wedge \neg p)$. This means that there is a path σ starting in π_{k+1} such that $(\mathfrak{M},\sigma_m) \not\models^\alpha (x \prec y \wedge \neg p)$, for any $m \in \mathbb{N}$. We define the path τ: $\tau_0 = \pi_k$ and $\tau_{n+1} = \sigma_n$, for any $n \in \mathbb{N}$; i.e., τ starts in π_k and then goes along σ. Since $(\mathfrak{M},\tau_0) \models^\alpha p$, we have $(\mathfrak{M},\tau_0) \not\models^\alpha (x \prec y \wedge \neg p)$ and from the same condition for all states in σ we obtain $(\mathfrak{M},\tau_n) \not\models^\alpha (x \prec y \wedge \neg p)$, for any $n \in \mathbb{N}$. But then $(\mathfrak{M},\tau_0) \not\models^\alpha \boldsymbol{AF}(x \prec y \wedge \neg p)$, i.e., $(\mathfrak{M},\pi_k) \not\models^\alpha \boldsymbol{AF}(x \prec y \wedge \neg p)$. Then, since $(\mathfrak{M},\pi_k) \models A_6$, we have $(\mathfrak{M},\pi_k) \not\models^\alpha x \prec y$, i.e., $a \not\prec_1^{\pi_k} b$. But, by induction hypothesis, it is not the case, hence the assumption $a \not\prec_1^{\pi_{k+1}} b$ is not true, and we have $a \prec_1^{\pi_{k+1}} b$.

Case $(\mathfrak{M},\pi_k) \not\models^\alpha p$. We have $(\mathfrak{M},\pi_k) \models^\alpha \neg p$ and with the same argumentation (using p instead of $\neg p$ and vice versa) we again obtain $a \prec_1^{\pi_{k+1}} b$.

Let $a \approx_1^{\pi_0} b$, for some $a, b \in D(\pi_0)$. The proof that $a \prec_1^{\pi_k} b$ proceeds by induction on k in the same way (with use of A_7 instead of A_6); we left the details to the reader. □

LEMMA 14. *Let $k \in \mathbb{N}$. Then there is $m \in \{0,\ldots,k\}$ such that $L^{\pi_k}(a_m)$ is true.*

PROOF proceeds by induction on k.

Let $k = 0$. Then we must prove that $L^{\pi_0}(a_0)$ is true; but we have $L^{\pi_0}(a_0)$ to be true by choosing of a_0.

Let the statement be true for k; we prove it for $k+1$. By induction hypothesis, we have $L^{\pi_k}(a_m)$, for some $m \in \{0,\ldots,k\}$. Let α be an interpretation of variables in π_k such that $\alpha(x) = a_m$, $\alpha(y) = a_{m+1}$. By Lemma 13, we have $a_m \prec_1^{\pi_k} a_{m+1}$ and hence $(\mathfrak{M},\pi_k) \models^\alpha \boldsymbol{AF}L(y)$ because $(\mathfrak{M},\pi_k) \models A_9''$.

Suppose $L^{\pi_{k+1}}(a_0),\ldots,L^{\pi_{k+1}}(a_{k+1})$ are not true. Then, in particular, $L^{\pi_{k+1}}(a_0),\ldots,L^{\pi_{k+1}}(a_{m+1})$ are not true and hence $(\mathfrak{M},\pi_{k+1}) \not\models^\alpha \exists x(x \preccurlyeq y \wedge L(x))$. Since $(\mathfrak{M},\pi_{k+1}) \models A_9''$, we obtain $(\mathfrak{M},\pi_{k+1}) \not\models^\alpha \boldsymbol{AF}L(y)$. This means that there is a path σ starting in π_{k+1} such that $(\mathfrak{M},\sigma_l) \not\models^\alpha L(y)$, for any $l \in \mathbb{N}$.

We define the path τ: $\tau_0 = \pi_k$ and $\tau_{n+1} = \sigma_n$, for any $n \in \mathbb{N}$.

We claim $(\mathfrak{M},\tau_n) \not\models^\alpha L(y)$, for any $n \in \mathbb{N}$. Indeed, it is the case for any $n \in \mathbb{N}^+$ by the choice of σ. As for $n = 0$, we have $\tau_0 = \pi_k$ and hence $(\mathfrak{M},\tau_0) \models^\alpha L(x)$. Supposing $(\mathfrak{M},\tau_0) \models^\alpha L(y)$, we obtain $(\mathfrak{M},\tau_n) \models^\alpha x \approx y$ because $(\mathfrak{M},\tau_0) \models A_8$. But then

$a_m \approx^{\pi_k} a_{m+1}$. From Lemmas 11 and 12 we obtain $a_m \not\prec^{\pi_k} a_{m+1}$, in particular, $a_m \not\prec_1^{\pi_k} a_{m+1}$, but this is impossible by Lemma 13. Thus $(\mathfrak{M}, \tau_0) \not\models^{\alpha} L(y)$.

Since $(\mathfrak{M}, \tau_n) \not\models^{\alpha} L(y)$ holds for any $n \in \mathbb{N}$, we obtain $(\mathfrak{M}, \tau_0) \not\models^{\alpha} \boldsymbol{AF}L(y)$, i. e., $(\mathfrak{M}, \pi_k) \not\models^{\alpha} \boldsymbol{AF}L(y)$, but this is not so. Hence our assumption is not true, therefore at least one of $L^{\pi_{k+1}}(a_0), \ldots, L^{\pi_{k+1}}(a_{k+1})$ is true. □

Now we are ready to prove a lemma that is similar to Lemmas 6 and 9. Recall that $\pi_0 = w^*$.

Lemma 15. *Let $b \in D(w^*)$. Then $b \approx^{w^*} a_m$, for some $m \in \mathbb{N}$.*

Proof. Since $(\mathfrak{M}, w^*) \models C$, in any path starting in w^* there is a world w such that $L^w(b)$ is true. In particular, this is true for the path π. Hence $L^{\pi_k}(b)$ is true, for some $k \in \mathbb{N}$. Then, by Lemma 15, there is $m \in \{0, \ldots, k\}$ such that $L^{\pi_k}(a_m)$ is true. Since $(\mathfrak{M}, \pi_k) \models A_8$, we have $b \approx^{\pi_k} a_m$. By Lemma 13, we obtain $b \approx^{\pi_0} a_m$, i. e., $b \approx^{w^*} a_m$. □

Proposition 7. *The structures $\langle D^{w^*}, \prec^{w^*} \rangle$ and $\langle \mathbb{N}, < \rangle$ are isomorphic.*

Proof. Follow to the proof of Proposition 3. □

Let φ be some closed classical first-order formula. We define $Emb''(\varphi)$:

$$Emb'(\varphi) \quad = \quad B'' \wedge C'' \wedge Congr(\varphi) \to T(\varphi).$$

Lemma 16. $\varphi \in \mathbf{QCL}_{fin} \iff Emb''(\varphi) \in \mathbf{QCTL}$.

Proof proceeds with the same argumentation as the proof of Lemma 7; the difference is in use Proposition 7 instead of Proposition 3. □

Corollary 4. *The logic* **QCTL** $\restriction \{\boldsymbol{AF}\}$ *is not recursively enumerable.*

9 Corollaries

First of all, now we are able to give a proof for Theorem 1. Indeed, let M be a set of modalities as in Theorem 1. If $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ or $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$ then the logic $\mathbf{QCTL} \restriction M$ is recursively enumerable and even finitely axiomatizable by Proposition 1. Suppose $M \not\subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ and $M \not\subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$. Then at least one of the following conditions holds:

- M contains $\boldsymbol{AX}$ or $\boldsymbol{EX}$ and M contains $\boldsymbol{AG}$ or $\boldsymbol{EF}$;
- M contains $\boldsymbol{EU}$;
- M contains $\boldsymbol{AU}$, or $\boldsymbol{AF}$, or $\boldsymbol{EG}$.

In every of these cases the logic $\mathbf{QCTL} \restriction M$ is not recursively enumerable by Corollaries 2, 3, and 4.

Now we give some other corollaries of Theorem 1 and its proof. Below let $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}, \boldsymbol{AG}, \boldsymbol{EG}, \boldsymbol{AF}, \boldsymbol{EF}, \boldsymbol{AU}, \boldsymbol{EU}\}$.

9.1 Finite axiomatizability

Note that any recursively enumerable fragment we consider here is also finitely axiomatizable; thus we have the following statement.

Corollary 5. *The logic* $\mathbf{QCTL} \restriction M$ *is finitely axiomatizable if and only if* $M \subseteq \{\boldsymbol{AX}, \boldsymbol{EX}\}$ *or* $M \subseteq \{\boldsymbol{AG}, \boldsymbol{EF}\}$.

We make a remark. From Theorem 1 and Corollary 5 it follows that a fragment $\mathbf{QCTL} \restriction M$ is recursively enumerable if and only if it is finitely axiomatizable. In general, recursive enumerability of a logic is not equivalent to its finite axiomatizability; it is equivalent just to its recursive axiomatizability. In our case such criterion is possible maybe because of finite number of fragments we consider.

Now let us turn to the other side of axiomatizability. We mean completeness.

9.2 Kripke completeness

Recall that a logic L is called *Kripke complete* if there is a class $\mathfrak{C}$ of (predicate) Kripke frames such that $L = \{\varphi : \mathfrak{C} \models \varphi\}$ where $\mathfrak{C} \models \varphi$ means φ is true in any frame in $\mathfrak{C}$.

Any fragment $\mathbf{QCTL} \upharpoonright M$ is Kripke complete by its definition; so, the question about Kripke completeness makes sense just for logics defined in other ways. Here we concern calculi.

We specify what we understand under a calculus. We assume a calculus to be defined by means of two sets: a set $\mathcal{A}$ of axioms and a set $\mathcal{R}$ of inference rules. These sets assumed to be recursively enumerable, moreover any inference rule assumed to be effective, i. e., as an algorithm computing the resulting formula. Of course, finite axiomatizations provides us with calculi.

Note that the set of all derivable formulas in a calculus (in our understanding) is recursively enumerable: to get an algorithm enumerating derivable formulas we may use an algorithm constructing a consequence of all derivations. The last algorithm exists because of algorithmic conditions we claim for the sets of axioms and inference rules.

Let $\mathcal{R}$ be a set of inference rules, X be a set of formulas. We denote by $\mathcal{C}_{\mathcal{R}}(X)$ the least set of formulas containing X and closed under inference rules in $\mathcal{R}$, generalization, modus ponens, and substitution (in an appropriate language). Let also $X \oplus_{\mathcal{R}} Y = \mathcal{C}_{\mathcal{R}}(X \cup Y)$.

Let L be a logic (a set of formulas). Let us call a set $\mathcal{R}$ of inference rules *L-admissible* if $\mathcal{C}_{\mathcal{R}}(L) \subseteq L$.

Let us call a set M of modalities *quite expressive* if the modalities in at least one of the sets $\{\boldsymbol{AX}, \boldsymbol{AG}\}$, $\{\boldsymbol{EU}\}$, $\{\boldsymbol{AF}\}$ can be expressed via the ones in M (maybe with use of the connectives).

Corollary 6. *Let M be a quite expressive set of modalities, S be a recursively enumerable set of propositional formulas in the language of* $\mathbf{CTL} \upharpoonright M$ *such that* $S \subseteq \mathbf{CTL} \upharpoonright M$*, and $\mathcal{R}$ be a* **QCTL***-admissible set of inference rules. Then the logic* $\mathbf{QCL} \oplus_{\mathcal{R}} S$ *is not Kripke complete.*

Proof. We give just a sketch of a proof. Let us use the denotation $QS = \mathbf{QCL} \oplus_{\mathcal{R}} S$. The logic QS may be viewed as a calculus with the axiom set $\mathbf{QCL} \cup S$, therefore it is recursively enumerable. Suppose it is Kripke complete. Then there is a class $\mathfrak{C}$ of Kripke frames such that $QS = \{\varphi : \mathfrak{C} \models \varphi\}$.

First of all observe that (modulo modality equivalence) at least one of the formulas $A \wedge B \wedge C$, $A' \wedge B \wedge C$, $B'' \wedge C''$ is a formula in the language of QS; this is so because M is quite expressive. Let us denote it by Φ. We claim

$$\varphi \in \mathbf{QCL}_{fin} \iff \Phi \wedge Congr(\varphi) \to T(\varphi) \in QS,$$

for any closed classical first-order formula φ.

Suppose $\varphi \notin \mathbf{QCL}_{fin}$. Then there is a classical model $\mathfrak{S} = \langle S, J\rangle$ such that S is finite and $\mathfrak{S} \not\models \varphi$; we may assume $S = \{0, \ldots, n\}$, for some $n \in \mathbb{N}$.

Observe that $\neg\Phi \notin QS$. Indeed, it follows from Propositions 2, 4, and 6 that $\neg\Phi \notin \mathbf{QCTL}$. But $\mathbf{QCL} \subset \mathbf{QCTL}$ and $S \subseteq \mathbf{CTL} \upharpoonright M \subset \mathbf{QCTL}$, therefore $QS \subseteq \mathbf{QCTL}$ because $\mathcal{R}$ is $\mathbf{QCTL}$-admissible.

Since $\neg\Phi \notin QS$, there is a frame $\mathfrak{F}(D) = \langle W, R, D\rangle$ such that $\mathfrak{F}(D) \in \mathfrak{C}$ (or $\mathfrak{F} \in \mathfrak{C}$ where $\mathfrak{F} = \langle W, R\rangle$) and $\mathfrak{F}(D) \not\models \neg\Phi$. Then there is a model $\mathfrak{M} = \langle W, R, D, I\rangle$ and a world $w^* \in W$ such that $(\mathfrak{M}, w^*) \not\models \neg\Phi$, i. e., $(\mathfrak{M}, w^*) \models \Phi$.

It follows from Propositions 6, 5, and 7 that the structures $\langle D^{w^*}, \prec^{w^*}\rangle$ and $\langle \mathbb{N}, <\rangle$ are isomorphic, i. e., there is an isomorphism $f : \mathbb{N} \to D^{w^*}$ for the structures. Let us consider the model $\mathfrak{M}' = \langle W, R, D, I'\rangle$ where I' is defined as I with the only difference for predicate letters occurring in φ: if P is m-ary letter occurring in φ and $b_1, \ldots, b_m \in D(w^*)$ then we put

$$\langle b_1, \ldots, b_m\rangle \in I'(w^*, P) \;\;\leftrightharpoons\;\; \text{there are } k_1, \ldots, k_m \in S \text{ such that } \langle k_1, \ldots, k_m\rangle \in J(P) \text{ and } b_i \in f(k_i), \text{ for any } i \in \{1, \ldots, m\}.$$

Then we obtain $(\mathfrak{M}', w^*) \not\models \Phi \wedge Congr(\varphi) \to T(\varphi)$, and hence $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin QS$; the details are left to the reader.

Now suppose $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin QS$. Since $QS \subseteq \mathbf{QCTL}$, we have $\Phi \wedge Congr(\varphi) \to T(\varphi) \notin \mathbf{QCTL}$, and hence $\varphi \notin \mathbf{QCL}_{fin}$ by Lemmas 6, 9, and 15.

But then QS is not recursively enumerable, and we have a contradiction. Hence QS is not Kripke complete. $\square$

9.3 The case of constant domains

Kripke frame $\mathfrak{F}(D) = \langle W, R, D\rangle$ is said to be a frame with *constant domains* if

$$wRw' \implies D(w) = D(w'),$$

for any $w, w' \in W$. The constant domain condition means that new elements do not appear when we go from one state to another, i. e., any element we may deal with in some future state is available in the current state, too.

Let $\mathbf{QCTL}^{cd}$ be a logic of serial Kripke frames with constant domains. Note that the propositional fragment of $\mathbf{QCTL}^{cd}$ is the logic **CTL**, i. e., $\mathbf{QCTL}^{cd}$ as well as **QCTL** is a conservative first-order extension of **CTL**. Here we put and answer the following question: do the presented results remain to be true if we replace **QCTL** with $\mathbf{QCTL}^{cd}$? And the answer is 'yes, of course'.

Indeed, it is sufficient to observe that the formulas $A \wedge B \wedge C$, $A' \wedge B \wedge C$, and $B'' \wedge C''$ are satisfiable in some models based on a serial frame with constant domains; see the proofs of Propositions 2, 4, and 6.

9.4 More simple fragments

We make remarks about fragments with restrictions on predicate letters. First of all, observe that it is possible to define $\approx$ and $\prec$ via L. So, for example in w^* (see Section 6)

$$\begin{aligned} x \approx y \quad &\text{means} \quad \boldsymbol{AG}(L(x) \leftrightarrow L(y)); \\ x \prec y \quad &\text{means} \quad \boldsymbol{AF}(L(x) \wedge \neg L(y) \wedge \boldsymbol{AF}L(y)). \end{aligned}$$

As for the $\mathbf{QCL}_{fin}$, its fragment with a binary predicate letter P is not recursively enumerable. It is known, see [8], that a binary predicate letter may be simulated with two unary letters: for example $P(x, y)$ may be simulated with $\boldsymbol{EX}(P'(x) \wedge P''(y))$. This means that three unary letters are sufficient to prove that the correspondent fragment of **QCTL** is not recursively enumerable.

Moreover even two unary letters are enough. It is known that the theory of finite models with symmetric irreflexive binary relation is not recursively enumerable; see [9]. But if a letter P corresponds to such relation then $P(x, y)$ may be simulated with $\boldsymbol{EX}((P'(x) \wedge \neg P'(y)) \vee (\neg P'(x) \wedge P'(y)))$.

As for number of individual variables, it seems to be truthful that three ones are enough. But in view of [7], we think that even two variables are enough.

10 Conclusion remarks

Let M be some quite expressive set of modalities. Suppose we are asked: why the fragment $\mathbf{QCTL} \upharpoonright M$ is not recursively enumerable? As a possible answer we may say that this is because one may simulate positive integers using the language of the fragment. But why one may do this?

To answer the question let us turn to the notion of path. A path π in a frame $\mathfrak{F} = \langle W, R \rangle$ is an infinite consequence of worlds $\pi_0, \pi_1, \pi_2, \ldots$ with the condition $w_k R w_{k+1}$, for any $k \in \mathbb{N}$. This means that any path is a map from $\mathbb{N}$ into W. Thus, we have positive integers as paths. Some modalities allow us to 'catch' them because of their definition in Kripke models.

But it is not the case for the pairs $\boldsymbol{AX}, \boldsymbol{EX}$ and $\boldsymbol{AG}, \boldsymbol{EF}$. This is so because these modalities are quite 'simple': accessibility relations for them are first-order definable. For example, for $\boldsymbol{AX}$ we need just seriality of R. The modality $\boldsymbol{AG}$ seems to be more complicated. It corresponds to the reflexive and transitive closure of R which is not first-order definable via R. And indeed, if we use $\boldsymbol{AX}$ and $\boldsymbol{AG}$ simultaneously then we are able to 'catch' positive integers. But if we use $\boldsymbol{AG}$ only then we 'lose' R and, in fact, we have just some reflexive and transitive accessibility relation.

11 Acknowledgements

We are thankful to all participant of the seminar on mathematical logic at Tver State University (Russia, Tver), especially to Alexander Chagrov and Lilia Chagrova, for their attention to details, interesting discussions and questions. We are also thankful to the researchers working at IRIT (Institut de Recherche en Informatique de Toulouse, Université Paul Sabatier, Toulouse, France) who participated in our seminar (in spite of holidays) for their interest to the subject; our special thanks to Philippe Balbiani and Sergei Soloviev for discussions at the seminar (and out of it), for good questions, and for a warm welcome.

References

[1] Boolos, G. S., Burgess, J. P., and R. C. Jeffrey, *Computability and Logic*, Cambridge University Press, 2007.

[2] Clarke, E. M., Grumberg, O., and D. A. Peled, *Model Cheking*, The MIT Press, 1999.

[3] Gabbay, D., Shehtman, V., and D. Skvortsov, *Quantification in Nonclassical Logic*. Preliminary draft.

[4] Hodkinson, I., and M. Reynolds, Temporal Logic, in P. Blackburn, J. Van Benthem and F. Wolter (eds.), *Handbook of Modal Logic*, Elsevier, 2007, pp. 655–720.

[5] Hodkinson, I., Wolter, F., and M. Zakharyaschev, Decidable fragments of first-order temporal logics, *Annals of Pure and Apllied Logic* 106:85–134, 2000.

[6] Hodkinson, I., Wolter, F., and M. Zakharyaschev, Decidable and undecidable fragments of first-order branching temporal logics, *Proceedings of the 17th Annual IEEE Symposium on Logic in Computer Science*, LICS, 2002, pp. 393–402.

[7] Kontchakov, R., Kurucz, A., and M. Zakharyaschev, Undecidability of First-Order Intuitionistic and Modal Logics with Two Variables, *Bulletin of Symbolic Logic* 11(3):428–438, 2005.

[8] Kripke, S. A., The Undecidability of Monadic Modal Quantification Theory, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 8:113–116, 1962.

[9] Nerode, A., and R. Shore, Second Order Logic and First Order Theories of Reducibility Orderings, in J. Barwise, H. J. Keisler, K. Kuner (eds.), *The Kleene Symposium*, North-Holland, Amsterdam, 1980, pp. 181–200.

[10] Prior, A. N., *Past, present and future*, Oxford University press, 1967.

[11] Trakhtenbrot, B. A., The Impossibility of an Algorithm for the Decidability Problem on Finite Classes, *Proceedings of the USSR Academy of Sciences* 70(4):569–572, 1950. (in Russian)

[12] Wolter, F., and M. Zakharyaschev, Decidable fragments of first-order modal logics, *Journal of Symbolic Logic* 66:1415–1438, 2001.