Equality of consequence relations in finite-valued logical matrices

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abstract. In this paper the procedure is presented that allows to determine in finite number of steps if consequence relations in two finite-valued logical matrices for propostional language L are equal.

Keywords: product of logical matrices, consequence relation, equality of matrices

In his paper 'A test for the equality of truth-tables' [2], J. Kalicki has described a general method for testing the equality of the classes of tautologies in different finite-valued matrices. Below I present a generalization of Kalicki's method which allows to test whether the consequence relations in two finite-valued logical matrices are equal.

First, the question of equality of consequence relations in two arbitrary matrices will be reduced to the question of the properties of a single matrix. This matrix will be obtained from initial matrices via the operation of product, but it will have four classes of truth-values instead of the standard two (designated and non-designated). On the basis of these four classes I will define several consequence relations. The properties that these relations display in the product matrix will define if two initial matrices are equal in terms of consequence relation. Then I will show that it is sufficient to consider a finite set of formulas to investigate the properties in question, and that therefore a finite number of steps is required to determine if consequence relations are equal in two finite-valued matrices.

Let us begin with some necessary definitions.

Definition 1. A logical matrix is a structure $\mathfrak{M} =< V, F, D >$, where V is the set of truth-values, F is a set of functions on V called *basic functions*, and D is a *designated* subset of V.

In this paper we will only consider the logical matrices where V is finite.

If for any n it is true that $\mathfrak{M}$ contains as much n-ary elements of F as there are n-ary connectives in some propositional language L, $\mathfrak{M}$ is a logical matrix for L. In that case we can establish a one-to-one correspondence between the elements of F and the connectives of L, and define a valuation of a formula in $\mathfrak{M}$.

Definition 2. A valuation v of formula A in $\mathfrak{M}$ is a homomorphism of L in $< V, F >$ such that

1. if A is a propositional variable, then $v(A) \in V$;
2. if $A_1, A_2, \cdots, A_n$ are formulas, and $\mathbb{C}$ is an n-ary connective of L, then $v(\mathbb{C}(A_1, A_2, \cdots, A_n)) = f^n(v(A_1), v(A_2), \cdots, v(A_n))$, where f^n is a function from F corresponding to $\mathbb{C}$.

The definition of consequence relation in $\mathfrak{M}$ is a standard one.

Definition 3. $\Gamma \vDash (\mathfrak{M})B$ iff there is no valuation v in $\mathfrak{M}$, such that $v[\Gamma] \subseteq D(\mathfrak{M})$ (i.e. every formula from Γ assumes a truth-value designated in $\mathfrak{M}$), and $v(A) \notin D(\mathfrak{M})$.

Let us denote as $C(\mathfrak{M})$ a set of ordered pairs $< \Gamma, B >$, such that Γ is a set of formulas, B is a formula, and $\Gamma \vDash (\mathfrak{M})B$. Now we will define the equality of consequence relations in two arbitary matrices for L.

Definition 4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the matrices for L. The consequence relations in $\mathfrak{A}$ and $\mathfrak{B}$ are equal iff $C(\mathfrak{A}) = C(\mathfrak{B})$.

Now we will make the transition from two matrices to one by applying the product operation. If $\mathfrak{A}$ and $\mathfrak{B}$ are the matrices for L, a one-to-one correspondence between the elemnts of their sets of basic functions can be established. This allows us to give the following definition.

Definition 5. A product of matrices $\mathfrak{A}$ and $\mathfrak{B}$ is a matrix $\mathfrak{C} = \mathfrak{A} \otimes \mathfrak{B}$, such that

- $V(\mathfrak{C})$ is a Cartesian product of $V(\mathfrak{A})$ and $V(\mathfrak{B})$;
- for each pair pair of mutually corresponding k-ary basic functions $f^k(x_1, x_2, \cdots, x_k)$ from $\mathfrak{A}$ and $g^k(y_1, y_2, \cdots, y_k)$ from $\mathfrak{B}$ there is one and only one basic operation h^k from $\mathfrak{C}$, and $h^k(< x_1, y_1 >, < x_2, y_2 >, \cdots, < x_k, y_k >) = < f^k(x_1, x_2, \cdots, x_k), g^k(y_1, y_2, \cdots, y_k) >$.

This is a standard product operation. However, the truth-values in $\mathfrak{C}$ will be divided into four classes[1]:

- $< x_i, y_j > \in \omega(\mathfrak{C})$ iff $x_i \in D(\mathfrak{A})$ and $y_j \in D(\mathfrak{B})$;
- $< x_i, y_j > \in \xi(\mathfrak{C})$ iff $x_i \in D(\mathfrak{A})$ and $y_j \notin D(\mathfrak{B})$;
- $< x_i, y_j > \in \xi'(\mathfrak{C})$ iff $x_i \notin D(\mathfrak{A})$ and $y_j \in D(\mathfrak{B})$;
- $< x_i, y_j > \in \phi(\mathfrak{C})$ iff $x_i \notin D(\mathfrak{A})$ and $y_j \notin D(\mathfrak{B})$.

I will now consider two definitions of consequense relation based on these four classes, $\models_\cup$ and $\models_\cap$.

DEFINITION 6. $\Gamma \models_\cup (\mathfrak{C})B$ iff there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \in \phi(\mathfrak{C})$.

LEMMA 1. $\Gamma \models_\cup (\mathfrak{C})B$ iff $\Gamma \models (\mathfrak{A})B$ or $\Gamma \models (\mathfrak{B})B$.

PROOF. *(i)* Let $\Gamma \models_\cup (\mathfrak{C})B$, and $\Gamma \not\models (\mathfrak{A})B$, and $\Gamma \not\models (\mathfrak{B})B$. Then there exists a valuation v^* in $\mathfrak{A}$, such that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$, and there exists a valuation u^* in $\mathfrak{B}$, such that $u^*[\Gamma] \subseteq D(\mathfrak{B})$ and $u^*(A) \notin D(\mathfrak{B})$. For every v and u there is a mapping w of the propositional variables of L on $V(\mathfrak{A}) \times V(\mathfrak{B})$, such that $w(p_k) =< v(p_k), u(p_k) >$, where p_k is a propositional variable. Obviously, every such w is a valuation in $\mathfrak{C}$. By definition of $\mathfrak{C}$, w^* obtained from v^* and u^* is such a valuation that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \phi(\mathfrak{C})$. That contradicts our assumption.

(ii) Let $\Gamma \not\models_\cup (\mathfrak{C})B$, and $\Gamma \models (\mathfrak{A})B$ or $\Gamma \models (\mathfrak{B})B$. Then there is a valuation w^* in $\mathfrak{C}$, such that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \phi(\mathfrak{C})$. For

[1]This is essentially a distribution introduced by Kalicki [2], but he only needed three classes, so elements of $\xi(\mathfrak{C})$ and $\xi'(\mathfrak{C})$ were assigned to the same class.

every valuation w in $\mathfrak{C}$ there is the following valuation v in $\mathfrak{A}$: if $w(p_k) =< x_i, y_j >$, then $v(p_k) = x_i$. By definition of $\mathfrak{C}$, v^* obtained this way from w^* is such a valuation in $\mathfrak{A}$ that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$. The reasoning for valuation u^* in $\mathfrak{B}$ is analogous, and leads to the contradiction. □

Definition 7. $\Gamma \vDash_\cap (\mathfrak{C})B$ iff all three of the following conditions are fulfilled:

- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$;
- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$;
- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

Lemma 2. $\Gamma \vDash_\cap (\mathfrak{C})B$ iff $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$.

Proof. *(i)* Let $\Gamma \vDash_\cap (\mathfrak{C})B$, and $\Gamma \nvDash (\mathfrak{A})B$, and $\Gamma \nvDash (\mathfrak{B})B$. The reasoning is analogous to the one in Lemma 1.

(ii) Let $\Gamma \vDash_\cap (\mathfrak{C})B$, and either $\Gamma \nvDash (\mathfrak{A})B$ or $\Gamma \nvDash (\mathfrak{B})B$. Suppose $\Gamma \nvDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$. Then there is a valuation v^* in $\mathfrak{A}$, such that $v^*[\Gamma] \subseteq D(\mathfrak{A})$ and $v^*(A) \notin D(\mathfrak{A})$. Now we have to consider two possibilities.

(ii.1) There is a valuation u^* in $\mathfrak{B}$, such that $u^*[\Gamma] \subseteq D(\mathfrak{B})$ and $u^*(A) \in D(\mathfrak{B})$. In this case, from v^* and u^* we can obtain a corresponding valuation w^* in $\mathfrak{C}$ (see Lemma 1), such that $w^*[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w^*(A) \in \xi'(\mathfrak{C})$. But then $\Gamma \nvDash_\cap (\mathfrak{C})B$, which contradicts our assumption.

(ii.2) For every valuation u in $\mathfrak{B}$, $u[\Gamma] \notin D(\mathfrak{B})$. Let u' be such a valuation that $u'[\Gamma] \notin D(\mathfrak{B})$, and $u'(A) \notin D(\mathfrak{B})$. The corresponding valuation w' in $\mathfrak{C}$ obtained from v^* and u' in the same way as in Lemma 1 will be such that $w'[\Gamma] \subseteq \xi(\mathfrak{C})$, and $w'(A) \in \phi(\mathfrak{C})$. Let u'' be such a valuation that $u''[\Gamma] \notin D(\mathfrak{B})$, and $u''(A) \in D(\mathfrak{B})$. The corresponding valuation w'' in $\mathfrak{C}$ obtained from v^* and u'' will be such that $w''[\Gamma] \subseteq \xi(\mathfrak{C})$, and $w''(A) \in \xi'(\mathfrak{C})$. Both cases lead us to the contradiction with the assumption that $\Gamma \vDash_\cap (\mathfrak{C})B$.

The reasoning for $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \nvDash (\mathfrak{B})B$ is analogous.

(iii) Let $\Gamma \nvDash_{\cap} (\mathfrak{C})B$, and $\Gamma \vDash (\mathfrak{A})B$, and $\Gamma \vDash (\mathfrak{B})B$. If $\Gamma \nvDash_{\cap} (\mathfrak{C})B$, three cases are possible:

(iii.1) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$;

(iii.2) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$;

(iii.3) There is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

The reasining for all three cases is the same. We obtain from w the corresponing valuations v in $\mathfrak{A}$ and u in $\mathfrak{B}$ in the same way as we did in Lemma 1. Due to the properties of w described in *(iii.1)*–*(iii.3)*, either v, or u, or both of them will be such that they will lead to the contradiction with the assumption that $\Gamma \vDash (\mathfrak{A})B$ and $\Gamma \vDash (\mathfrak{B})B$. □

From Lemma 1 we have that $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{A}) \cup C(\mathfrak{B})$. From Lemma 2 we have that $C(\mathfrak{C}, \vDash_{\cap}) = C(\mathfrak{A}) \cap C(\mathfrak{B})$. Also, we have that $C(\mathfrak{A}) = C(\mathfrak{B})$ iff $C(\mathfrak{A}) \cup C(\mathfrak{B}) = C(\mathfrak{A}) \cap C(\mathfrak{B})$. Therefore, $C(\mathfrak{A}) = C(\mathfrak{B})$ iff $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{C}, \vDash_{\cap})$.

Now let us consider another consequence relation.

DEFINITION 8. $\Gamma \vDash^{*} (\mathfrak{C})B$ iff either

- there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C})$,
- and there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$,
- and there is no valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(A) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$,
- or there is a valuation w in $\mathfrak{C}$, such that $w[\Gamma] \subseteq \omega(\mathfrak{C})$, and $w(A) \in \phi(\mathfrak{C})$.

LEMMA 3. $C(\mathfrak{C}, \vDash_{\cup}) = C(\mathfrak{C}, \vDash_{\cap})$ iff $\Gamma \vDash^{*} (\mathfrak{C})B$ for each set of formulas Γ and each formula B.

Proof. If $C(\mathfrak{C}, \models_\cup) = C(\mathfrak{C}, \models_\cap)$, for each Γ and B it is true that either $\Gamma \models_\cap (\mathfrak{C})B$ or $\Gamma \not\models_\cup (\mathfrak{C})B$. Both cases lead to $\Gamma \models^* (\mathfrak{C})B$. Now let us assume that $\Gamma \models^* (\mathfrak{C})B$ for some arbitrary Γ and B. Then *(i)* for every evaluation w in $\mathfrak{C}$, if $w[\Gamma] \subseteq \omega(\mathfrak{C})$ then $w(A) \in \omega(\mathfrak{C})$, if $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, then $w(A) \in \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, if $w[\Gamma] \subseteq \omega(\mathfrak{C}) \cap \xi'(\mathfrak{C})$, then $w(A) \in \omega(\mathfrak{C}) \cap \xi'(\mathfrak{C})$, or *(ii)* there is at least one valuation in $\mathfrak{C}$, such that all formulas from Γ assume a truth value from $\omega(\mathfrak{C})$, and B assumes a value from $\phi(\mathfrak{C})$. In the first case $\Gamma \models_\cap (\mathfrak{C})B$. In the second case $\Gamma \not\models_\cup (\mathfrak{C})B$. Therefore $C(\mathfrak{C}, \models_\cup) = C(\mathfrak{C}, \models_\cap)$. □

Below, the number of formulas that need to be considered will be narrowed down to a finite set. I will use the method proposed by J. Kalicki in [1] with necessary modifications.

Lemma 4. For each matrix $\mathfrak{C}_m$, where m is the number of the elements of $V(\mathfrak{C})$, the following is true: if for each pair Γ and B that contains $i \leq m$ different variables $\Gamma \models^* (\mathfrak{C}_m)B$, then for each pair Δ and E that contains $m + t (t = 0, 1, \cdots)$ different variables $\Delta \models^* (\mathfrak{C}_m)E$.

Proof. Let us use the induction by t. For $t = 0$ it is obvious that for each Γ and B that contains $i \leq m$ different variables $\Gamma \models^* (\mathfrak{C}_m)B$, then for each pair$\Delta$ and E that contains m different variables $\Delta \models^* (\mathfrak{C}_m)E$.

Let us assume that the theorem is true for $t \leq k$ and prove it for $t = k + 1$. Let there exist Δ and E that contain $m + k + 1$ different variables, and $\Delta \not\models^* (\mathfrak{C}_m)E$. Then there exists a valuation w_0 in $\mathfrak{C}_m$ that maps the variables $p_1, p_2, \cdots, p_{m+k+1}$ on values $x_1, x_2, \cdots, x_{m+k+1}$ respectively, such that either *(i)* $w_0[\Delta] \subseteq \omega(\mathfrak{C})$, and $w_0(E) \notin \omega(\mathfrak{C})$, or *(ii)* $w_0[\Delta] \subseteq \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, and $w(E) \notin \omega(\mathfrak{C}) \cup \xi(\mathfrak{C})$, or *(iii)* $w_0[\Delta] \subseteq \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$, and $w(E) \notin \omega(\mathfrak{C}) \cup \xi'(\mathfrak{C})$.

Let us consider (i). Due to the fact that in $\mathfrak{C}_m$ there is m different truth-values in total, there will be at least two $i_1 \neq i_2$ among $i = 1, 2, \cdots, m + k + 1$, such that $x_{i_1} = x_{i_2}$. Now let us consider Δ' and E', obtained from Δ and D by replacement of all instances of p_{i_2} with p_{i_1}. It is clear that $w_0[\Delta'] \subseteq \omega(\mathfrak{C}_m)$ and $w_0(E') \notin \omega(\mathfrak{C}_m)$. Because Δ' and E' contain $m + k$ different variables, according to the inductive assumption, $\Delta' \models^* (\mathfrak{C}_m)E'$. Therefore, there exists a valuation w^* in

$\mathfrak{C}_m$, which maps the variables $p_1, p_2, \cdots, p_{i_2-1}, p_{i_2+1}, \cdots, p_{m+k+1}$ on the values $y_1, y_2, \cdots, y_{i_2-1}, y_{i_2+1}, \cdots, y_{m+k+1}$ respectively, such that $w^*[\Delta'] \subseteq \omega(\mathfrak{C}_m)$ and $w^*(E') \in \phi(\mathfrak{C}_m)$. In this case we can construct a valuation w^{**}, which maps the variables $p_1, p_2, \cdots, p_{m+k+1}$ on the values $y_1, y_2, \cdots, y_{i_2-1}, y_{i_1}, y_{i_2+1}, \cdots, y_{m+k+1}$ respectively. It is clear that $w^{**}[\Delta] \subseteq \omega(\mathfrak{C}_m)$ and $w^{**}(E) \in \phi(\mathfrak{C}_m)$. But then $\Delta \models^* (\mathfrak{C}_m)E$, which contradicts our assumption.

The reasoning for (ii) and (iii) is analogous. □

For m different variables there is $k = m^m$ different valuations $v_1, v_2, \cdots, v_k$ in $\mathfrak{C}_m$. We can assign to each variable $p_i (1 \leq i \leq m)$ a unique value-sequence $|p_i| =< x_1, x_2, \cdots, x_k >$, where $x_l = v_l(p_i)(1 \leq l \leq k)$.

Now let us construct the following sequence of the classes of formulas:

- The elements of CL_0 are the variables $p_1, p_2, \cdots, p_m$ exclusively;
- to a class CL_{t+1} belong all formulas that can be constructed by means of one connective, an element of class CL_t, and (if needed) elements of $CL_{n \leq t}$.

For each formula B from CL_n we can calculate the corresponding value-sequence $|B| =< y_1, y_2, \cdots, y_k >$, where $y_j (1 \leq j \leq k)$ is obtained from j-th elements of sequences assigned to the variables included in B. Let us denote the set of value-sequences for elements of Cl_n as $|Cl_n|$. Because the sequences in question consist of k elements, and the number of truth-values equals m, in total there is m^k possible sequences. Therefore, there is a finite $n_0 \leq m^k$, such that $|CL_{n_0}|$ contains no value-sequence which is not also the element of some $|CL_{n<n_0}|$.

Lemma 5. The value-sequence of any formula $B \in CL_{n>n_0}$ is identical to some element of $|CL_{n<n_0}|$.

Proof. Let $B \in CL_{n_0+1}$. By definition of CL_{n_0+1}, formula B consists of the main connective, at least one formula from CL_{n_0}, and probably elements of $CL_{n_i<n_0}$. By definition of n_0, each value-sequence from $|CL_{n_0}|$ is also present in some $|CL_{n_j<n_0}|$. Therefore, by definition of $|CL|$, there is a set $|CL_{max(i,j)+1}|$, which contains the value-sequence identical to $|B|$. Because $n_i < n_0$ and $n_j < n_0$, we have that $max(n_i, n_j) + 1 \leq n_0$, $|B| \in |CL_{n\leq n_0}|$. From that, according to the definition of n_0, we obtain that $|B| \in |CL_{n<n_0}|$. The theorem is proved for CL_{n_0+1}. The generalization for $CL_{n>n_0}$ is obvious. □

So the set $|CL_1|\cup|CL_2|\cup\cdots\cup|CL_{n_0}|$ contains all value-sequences possible in $\mathfrak{C}_m$ for formulas that contain no more than m different variables. From this fact and Lemma 4 it follows that $\Gamma \vDash^* (\mathfrak{C}_m)B$ for each Γ and B iff $\Delta \vDash^* (\mathfrak{C}_m)E$ for every Δ and E that consist exclusively of the elemnts of $CL_1 \cup CL_2 \cup \cdots \cup CL_{n_0}$.

This concludes the construction of the procedure for testing if $C(\mathfrak{A}) = C(\mathfrak{B})$ for two arbitrary finite-valued matrices $\mathfrak{A}$ and $\mathfrak{B}$ for some propositional language L.

References

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[2] Kalicki, J., A test for the equality of truth-tables, *Journal of Symbolic Logic* 17:161–163, 1952.