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EFFECTS IN QUANTUM LOGIC OF OBSERVABLES*

Abstract. *In the paper a modal and bimodal extension of quantum logic of observables QLO is proposed. The former allows to obtain the syntactic counterpart of D.Mundici's result on embedding of C*-algebra into an MV-algebra while the latter has as its algebraic counterpart quantum MV-algebra of R.Giumtini. The soundness and completeness of both extensions is proved in respect to the set-theoretical semantics developed early for QLO.*

1. Introduction

In [4] D.Mundici shown that every approximately dimensional *C**-algebra with lattice dimensional group can be embedded into a countable MV algebra. Since such an MV algebra is also a Lindenbaum algebra of Łukasiewicz infinite-valued calculus $Ł_\infty$ (the notion of MV algebra was introduced by C.C.Chang in order to provide an algebraic proof of the completeness theorem for $Ł_\infty$) then this result would be treated as a tool for considering properties of quantum systems in the framework of $Ł_\infty$. Needless to say that from the physical point of view in this case we ought to consider an elements of MV algebra as a class of operators whose spectrum is contained in the real interval [0,1].

But the lack of developed interpretation of such operators forces us to approach those as so-called *effects* of a Hilbert space which are bounded linear operators such that for an every effect E and for all density operators D, $0 \leq \mathrm{Tr}(DE) \leq 1$ (Born probability). It was shown by R.Giuntini [2] that the class of all effects of any Hilbert space turns out to be an instance of an algebraic structure called *quantum MV algebras*. Those retain some important properties of MV algebras, while violating the crucial axiom of MV algebras: the so-called Łukasiewicz axiom. Quantum MV algebras represent non-idempotent extension of orthomodular lattices just as MV algebras represent non-idempotent extensions of Boolean algebras.

Thus, in case of transferring Mundici's method onto quantum MV algebra of effects we can interpret those as determining a kind of Born probabilities for quantum observables represented by operators in Hilbert space. In fact, those probabilities would be considered as probabilities for observables to have as the result of measurement a certain

* The work is supported by RFBR grant No 01-06-80006.

magnitude contained in the real line of numbers (as projectors in Hilbert space would be regarding as "yes-no" answering the same question).

2. Quantum Logic of Observables

We obtain the syntactic version of Mundici's result if we have recourse to the so-called quantum logic of observables QLO [5]. QLO is axiomatized by means of the following axiom schemes and the rules:

Ax1. $A \Rightarrow A$;

Ax2. $A \Leftrightarrow \neg\neg A$;

Ax3. $A\wedge(B\wedge C) \Leftrightarrow (A\wedge B)\wedge C$;
Ax4. $A\vee(B\vee C) \Leftrightarrow (A\vee B)\vee C$;
Ax5. $A\wedge(B\vee C) \Leftrightarrow (A\wedge B)\vee(A\wedge C)$;
Ax6. $\neg(A\vee\neg A) \Rightarrow B\wedge B$;
Ax7. $A\wedge\neg A \Rightarrow \neg(B\vee\neg B)$;
Ax8. $1\wedge A \Leftrightarrow A$
Ax9. $J_0 A \Leftrightarrow \neg(B\vee\neg B)$;
Ax10. $J_1 A \Leftrightarrow A$;
Ax11. $J_\alpha(A\wedge B) \Leftrightarrow J_\alpha B\wedge A$;
Ax12. $J_\alpha(A\vee B) \Leftrightarrow J_\alpha B\vee J_\alpha A$;
Ax13. $\neg J_\alpha A\Leftrightarrow J_\alpha\neg A$
Ax14. $J_{\alpha+\beta}A \Leftrightarrow J_\alpha A\vee J_\beta A$;
Ax15. $J_{\alpha\beta}A \Leftrightarrow J_\alpha J_\beta A$
Ax16. $\neg[(A\wedge B)\vee\neg(B\wedge A)]^2 \Leftrightarrow (A\vee\neg A)^2\wedge(B\vee\neg B)^2$ (A^2 means $A\wedge A$).

Rx1. $\dfrac{A \Rightarrow B}{\neg B \Rightarrow \neg A}$

Rx2. $\dfrac{A \Rightarrow B}{J_{|\alpha|}A \Rightarrow J_{|\alpha|}B}$

Rx3. $\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$

Rx4. $\dfrac{A \Rightarrow B \quad C\Rightarrow D}{A\vee C \Rightarrow B\vee D}$

Rx5. $\dfrac{A\wedge A \Rightarrow B \quad C\wedge C \Rightarrow D}{(A\wedge A) \wedge (C\wedge C) \Rightarrow B\wedge D}$

Here $A \Rightarrow B$ means $\langle A,B\rangle\in$L, where L is some logics, truth-value of $J_\alpha A$ is calculated as the result of multiplying truth-value of A on α being a real number.

Let Γ be a non-empty set of wff. A wff A is said to be QLO-*derivable* from Γ, $\Gamma\Rightarrow A$, if there exist $B_1,\ldots,B_n\in \Gamma$ such that

(a) either $B_1\vee\ldots\vee B_n\Rightarrow A$;
(b) or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow A$;
(c) or $J_{|\alpha i|}B_i\Rightarrow A$, $i = 1,2,\ldots,n$.

If A is QLO-derivable from $\neg(A\vee\neg A)$ then A is QLO-*derivable* or is a QLO–*theorem* which writes $\Rightarrow A$. Γ is QLO–*consistent* if there is at least one wff not QLO–derivable from Γ, and QLO–*inconsistent* otherwise (it can be shown that Γ is QLO-consistent iff for no A do we have both $\Gamma\Rightarrow A$ and $\Gamma\Rightarrow\neg A$). Γ is QLO–*full* iff it is QLO–consistent and closed under $\vee$, $\wedge$, J and QLO–derivability, i.e. iff

(1) for some wff A, not $\Gamma\Rightarrow A$;
(2) if $A\in\Gamma$ and $A\Rightarrow B$, then $B\in\Gamma$:
(3) $A,B\in\Gamma$ implies $A\wedge B$, $A\vee B\in\Gamma$;
(4) $A\in\Gamma$ implies $J_{|\alpha|}A\in\Gamma$.

If $x\subseteq \Phi$ (where Φ is a set of wff) is QLO–full then
(i) $x\Rightarrow A$ iff $A\in x$;
(ii) $\neg(A\vee\neg A)\in x$, for all wff A.

QLO-full sets and QLO-derivability are linking with the following version of Lindenbaum's Lemma:

$\Gamma\Rightarrow A$ iff A belongs to QLO-full extension of Γ.

It is proved that if x is QLO-full and $\neg A\notin x,$ then there exists a *QLO*-full set y such that $A\in y,$ and for all B, either $\neg B\notin x$ or $B\notin y.$

QLO have some peculiarities featuring quantum orthologic. Both in QLO and quantum orthologic the proof of Lindenbaum's Lemma does not require such power tools as, for example, Zorn's Lemma, which was in case of orthologic regarded as unprecedented for logical systems. As to the QLO-full sets, then from topological point of view they are, in fact, proper filters and not the ultrafilters. This, in turn, leads that for both quantum orthologic and QLO there is not need in some version of an axiom of choice which is required to prove an existence of ultrafilters.

It is easy to see that an algebra corresponding to QLO be an algebra of observables satisfying the axioms of algebraic approach in [1]. If we define an equivalency of formulas A and B, $A \sim B$ as $+ A \Leftrightarrow B$ then denoting the set $A/_\sim$ as $[A]$ we obtain

$[A]+[B] = [A\vee B]$,
$[A]\circ[B] = [A\wedge B]$,
$-[A]= [\neg A]$, $0 = [\neg(A\vee\neg A)]$,
$1 = [1]$, $\alpha[A] = [J_{|\alpha|}A]$.

A structure $\boldsymbol{F}= \langle F,+,\circ,-,\alpha,0,1\rangle$ (where $F = \{P/_\sim: P$ is a formula$\}$, $\alpha\in\mathbf{R}$) is an algebra (of observables) while $\boldsymbol{E} = \langle F,+,-,\alpha,0\rangle$ be a vector (linear) space, 0 is a unit relative to $+$, and 1 is a unit relative to $\circ$.

3. Modal Quantum Logic of Effects

Let us modify our formulation of QLO by replacing Ax1 with

Ax1′. $\neg(A\vee B) \Leftrightarrow \neg A\vee\neg B$

The following theorems of QLO will be used in the sequel:

Bx1. $\neg(A\vee\neg A) \vee B \Leftrightarrow B$

It is easy to see that this modification does not lead to any change of QLO. As to the Ax1 then $A \Rightarrow A$ can be proved from Ax2 by means of Rx3. Bx1 is proved by means of Ax9, Ax10, Ax14.

To introduce effects into QLO we enrich the language of QLO with a unary operator Q and axiomatics of QLO with the following axiom schemes and the rule:

Ax17. $\mathrm{Q}A\Leftrightarrow \mathrm{QQ}A$
Ax18. $\mathrm{Q}\neg A \Leftrightarrow 1\vee\neg\mathrm{Q}A$
Ax19. $\mathrm{Q}1 \Leftrightarrow 1$
Ax20. $\mathrm{Q}(A\vee B) \Leftrightarrow (\mathrm{Q}A\vee \mathrm{Q}B)$
Ax21. $\neg(B\vee\neg B) \Rightarrow \mathrm{Q}A \Rightarrow 1$
Ax22. $1 \Leftrightarrow 1\vee\mathrm{Q}A$

Rx6. $\dfrac{A \Rightarrow B}{\mathrm{Q}A \Rightarrow \mathrm{Q}B}$

Let us denote the system QLO + {Ax17-Ax23, Rx6} as QLO-MV (with Ax1′). In order to prove that QLO-MV really describes the effects let us firstly recall the algebraic structure responsible for those. According to P.Mangani [3] MV algebras can be defined in the following way:

(MV1) $(a\oplus b) \oplus c = a\oplus(b \oplus c)$
(MV2) $a\oplus 0 = a$
(MV3) $a\oplus b = b\oplus a$
(MV4) $a\oplus 1 = 1$
(MV5) $(a^*)^* = a$
(MV6) $0^* = 1$
(MV7) $a\oplus a^* = 1$
(MV8) $(a^*\oplus b)^*\oplus b = (a\oplus b^*)^*\oplus a$ (*Łukasiewicz axiom*)

As in QLO we define $[A] \oplus [B] = [\mathrm{Q}A\vee \mathrm{Q}B]$ and $[A]^* = [\neg \mathrm{Q}A]$.

Theorem 3.1. *A structure* $\mathsf{F} = \langle F, \oplus, ^*, 0, 1\rangle$ *where* $F = \{P/_\sim$: *P is a formula prefixed with Q*}. $0 = [\neg(A\vee\neg A)]$, $1 = [1]$ *is an MV algebra*.

Proof. Associativity of ⊕ for (MV1) follows from the definition of ⊕ and associativity of ∨ in QLO as well as commutativity for (MV3). (MV2) is fulfilled since $[A]\oplus 0$ is defined by $\mathrm{Q}A\vee \mathrm{Q}\neg(A\vee\neg A) \Leftrightarrow \mathrm{Q}(A\vee\neg(A\vee\neg A))$ and then by Bx1 it will be equivalent to $\mathrm{Q}A$ which under the definition of P gives us $[A]$. In case of ($MV4$) we have $QA\vee Q1 \Leftrightarrow QA\vee 1$ by Ax19. As to ($MV5$) then $[A]^{**}$ is determined by $Q\neg Q\neg A$ and by Ax18, Ax17 it gives us $Q\neg Q\neg A \Leftrightarrow 1\vee\neg Q\neg A \Leftrightarrow 1\vee\neg(Q\neg A) \Leftrightarrow 1\vee\neg 1\vee QA$. But we obtain $1\vee\neg 1 \Leftrightarrow \neg\neg 1\vee\neg\neg\neg 1 \Leftrightarrow \neg(\neg 1\vee\neg\neg 1) \Leftrightarrow \neg(1\vee\neg 1)$ with the help of Ax2, Ax1'. So, by Bx1 we obtain $(1\vee\neg 1)\vee QA \Leftrightarrow QA$. ($MV6$) follows from $Q\neg\neg(A\vee\neg A) \Leftrightarrow Q(A\vee\neg A) \Leftrightarrow QA\vee Q\neg A \Leftrightarrow QA\vee 1\vee\neg QA \Leftrightarrow 1$.

In order to obtain ($MV7$) we have $QA\vee Q\neg A$ by the definition and Ax17. Then like in case of ($MV6$) we get $QA\vee Q\neg A \Leftrightarrow 1$.

In case of Łukasiewicz axiom for the left part we have $Q\neg(Q\neg A\vee QB)\vee QB$ by the definitions and Ax17. Now by Ax18 and Ax17 we obtain $Q\neg(Q\neg A\vee QB)\vee QB \Leftrightarrow Q\neg Q\neg A\vee Q\neg QB\vee QB \Leftrightarrow 1\vee\neg Q\neg A\vee Q\neg QB\vee QB \Leftrightarrow 1\vee\neg(1\vee\neg QA)\vee 1\vee\neg QB\vee QB$. By Ax1', Ax2, Ax1' we have $1\vee\neg(1\vee\neg QA)\vee 1\vee\neg QB\vee QB \Leftrightarrow QA\vee 1 \Leftrightarrow 1$. For the right part we likewise obtain $Q\neg(Q\neg B\vee QA)\vee QA \Leftrightarrow QB\vee 1 \Leftrightarrow 1$ and this determines that Łukasiewicz axiom will be satisfied. □

In the sequel under wff we always mean a wff prefixed with Q.

Definition 3.2. Let Γ be a non-empty set of wff. A wff A is said to be QLO-MV-*derivable* from Γ, $\Gamma\Rightarrow A$, if there exist $B_1,\ldots,B_n\in\Gamma$ such that

(a) either $B_1\vee\ldots\vee B_n\Rightarrow A$;

(b) or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow A$;

(c) or $J_{|\alpha_i|}B_i\Rightarrow A$, $i = 1,2,\ldots,n$;

(d) or $QB_i\Rightarrow A$, $i = 1,2,\ldots,n$.

If A is QLO-MV-derivable from 1 then A is QLO-MV-*derivable* or is a QLO-MV-*theorem* which writes $\Rightarrow A$. Γ is QLO-MV-*consistent* if there is at least one wff not QLO-MV-derivable from Γ, and QLO-MV-*inconsistent* otherwise (it can be shown that Γ is QLO-MV-consistent iff for no A do we have both $\Gamma\Rightarrow A$ and $\Gamma\Rightarrow Q\neg A$). Γ is QLO-MV-*full* iff it is QLO-MV-consistent and closed under $\vee$, $\wedge$, J, Q and QLO-MV-derivability, i.e. iff

(1) for some wff A, not $\Gamma\Rightarrow A$;

(2) if $A\in\Gamma$ and $A\Rightarrow B$, then $B\in\Gamma$:

(3) $A,B\in\Gamma$ implies $A\wedge B$, $A\vee B\in\Gamma$;

(4) $A\in\Gamma$ implies $J_{|\alpha|}A\in\Gamma$;

(5) $A\in\Gamma$ implies $QA\in\Gamma$.

Lemma 3.3. *If* $x\subseteq\Phi$ (*where* Φ *is a set of wff*) *is QLO-MV-full, then*

(i) $x\Rightarrow A$ *iff* $A\in x$;

(ii) $Q\neg(A\vee\neg A)\in x$, *for all wffs A prefixed with* Q.

Proof. (i) Since in QLO-MV $A\Rightarrow A$, sufficiency follows from the definition of QLO-MV-deivability. Necessity follows from 3.2(2), (3), (4), (5).

(ii) By definition x is non-empty, thus there exists $B\in x$. But by 3.2(4) $J_0B\in x$, so by Ax9and 3.2(2) we have $\neg(A\vee\neg A)\in x$. The result follows under Ax8 and 3.2(2). □

QLO-MV-full sets and QLO-MV-derivability are connected with the following version of Lindenbaum's Lemma :

Theorem 3.4. $\Gamma\Rightarrow A$ *iff* A *belongs to every QLO-MV-full extension of* Γ.

Proof. If $\Gamma\Rightarrow A$ then there are $B_1,\ldots,B_n\in\Gamma$ such that either $B_1\vee\ldots\vee B_n\Rightarrow A$ or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow A$, or $J_{|\alpha_i|}B_i\Rightarrow A$ or $QB_i\Rightarrow A$ $(1\leq i\leq n)$. If x is QLO-MV-full and $\Gamma\subseteq x$, then $B_1,\ldots,B_n\in x$. Applying 3.2(3),(4),(5) and then 3.2(2), we obtain $A\in x$.

The other way round, suppose A is not QLO-MV-derivable from Γ. We put $x=\{B:\Gamma\Rightarrow B\}$. By Ax1 we have $\Gamma\subseteq x$, and by hypothesis $A\notin x$. The proof will be accomplished if we can show that x is QLO-MV-full. Suppose $B\in x$ and $B\Rightarrow C$, then there exist $B_1,\ldots,B_n\in\Gamma$ such that either $B_1\vee\ldots\vee B_n\Rightarrow B$, or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow B$, or $J_{|\alpha_i|}B_i\Rightarrow B$, or $QB_i\Rightarrow A$ $(1\leq i\leq n)$. So by Rx3 we obtain either $B_1\vee\ldots\vee B_n\Rightarrow C$, or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow C$, or $J_{|\alpha_i|}B_i\Rightarrow C$, or $QB_i\Rightarrow A$, hence, $\Gamma\Rightarrow C$, i.e. $C\in x$.

On the other side, if $B,C\in x$, then there exist $B_1,\ldots,B_n,C_1,\ldots,C_n\in\Gamma$, such that $B_1\vee\ldots\vee B_n\Rightarrow B$ and $C_1\vee\ldots\vee C_n\Rightarrow C$. Then by Rx4 we obtain $B_1\vee\ldots\vee B_n\vee C_1\vee\ldots\vee C_n\Rightarrow B\vee C$. So we have $\Gamma\Rightarrow B\vee C$ and thus $B\vee C\in x$.

Furthermore, if $B\in x$, then there exists $B'\in\Gamma$ such that $J_{|\alpha'|}B'\Rightarrow B$. But by Rx2 $J_{|\beta|}J_{|\alpha'|}B'\Rightarrow J_{|\beta|}B$ and by Ax15 we obtain $J_{|\beta\alpha'|}B'\Rightarrow J_{|\beta|}B$, thus, $\Gamma\Rightarrow J_{|\beta|}B\in x$.

Again, let $B,C\in x$. Then there exist $B_1,\ldots,B_n$, $C_1,\ldots,C_n\in\Gamma$ such that $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow B$ and $(C_1\wedge\ldots\wedge C_n)\wedge(C_n\wedge\ldots\wedge C_1)\Rightarrow C$. But then by Rx5 we obtain $(B_1\wedge\ldots\wedge B_n\wedge C_1\wedge\ldots\wedge C_n)\wedge(C_n\wedge\ldots\wedge C_1\wedge B_n\wedge\ldots\wedge B_1)\Rightarrow B\wedge C$. Hence, $\Gamma\Rightarrow B\wedge C$ and $B\wedge C\in x$.

If $B\in x$, then there exists $B'\in\Gamma$ such that $QB'\Rightarrow B$. Since by Rx6 $QQB'\Rightarrow QB$ then under Ax17 we have $QB'\Rightarrow QB$. Thus, $\Gamma\Rightarrow QB\in x$.

This shows that x is closed under QLO-MV-derivability, conjunction, disjunction, J- and Q-operators. Since $A\notin x$ then A is not QLO-MV-derivable from x, therefore x is QLO-MV-consistent. □

Theorem 3.5. *If* x *is QLO-MV-full and* $Q\neg A\notin x$, *then there exists QLO-MV-full set* y *such that* $A\in y$, *and for all* B, *either* $Q\neg B\notin x$ *or* $B\notin y$.

Proof. Let $y=\{B: A\Rightarrow B\}$. By Ax1 $A\in y$. Now let $\neg B\in x$. Then $B\notin y$, or else $A\Rightarrow B$, whence $\neg B\Rightarrow\neg A$ by Rx1, and so, in turn, by Rx6 and by 3.2(2), $Q\neg A\in x$ contrary to hypothesis. By 3.2(ii) we have $Q\neg(A\vee\neg A)\in x$. According to what we just proved, it follows that $A\vee\neg A\notin y$. Proceeding in a similar manner to 3.4 we can show that y is closed under $\vee,\wedge;J$,Q and QLO-MV-derivability. Then since $A\vee\neg A$ is not QLO-MV-derivable from y, i.e. y is QLO-MV-consistent, y be QLO-MV-full as required. □

Thus, in turn, for QLO-MV there is also not need in some version of an axiom of choice which is required to prove an existence of ultrafilters.

4. Semantics of QLO-MV

Since our logic is an extension of QLO then we adduce main definitions of QLO-semantics modifying them as may be necessary for QLO-MV.

Definition 4.1. QLO-MV-model is a 4-tuple $M = \langle X,\perp,\xi,v\rangle$, where

(a) X is a non-empty set;

(b) $\perp$ is an orthogonality relation on X;

(c) ξ is a non-empty collection of $\perp$-closed subsets of X closed under set-theoretic intersection and the operation * (Y* is defined as {x: $x\perp y$});

(d) v is a function assigning to each propositional variable and formula of QLO-MV recursively in every point (every element) of X a real number, i.e. v: $(S\cup\Phi) \times X \to \mathbf{R}$ where S is a set of propositional variables and Φ is a set of wffs.

Denoting the set $\{x\in X: v(A,x) = a\}$ as $\|A\|_a$ we define recursively the value of a wff in a QLO-MV-model as follows:

(1) $\|p_i\|_a = \{x\in X: v(p_i,x) = a\}\in\xi$;

(2) $\|A\vee B\|_a = \{x\in X: x\in\|A\|_b\cap\|B\|_c \;\&\; a = b + c\}$;

(3) $\|A\wedge B\|_a = \{x\in X: x\in\|A\|_b\cap\|B\|_c \;\&\; a = bc\}$;

(4) $\|\neg A\|_a = \{x\in X: x\perp\|A\|_{-a} \;\&\; v(\neg A,x) = a \}$;

(5) $\|J_{|\alpha|}A\|_a = \{x\in X: x\in\|A\|_b \;\&\; a = \alpha b\}$;

(6) $\|1\|_1 = X$ т.е. $v(1,x) = 1$ for all $x\in X$;

(7) $\|QA\|_a = \{x\in X: x\in\|A\|_b \;\&\; a = q(b)\}$ where q: $v \to [0,1]$ such that

   (i) $q(q(v(A))) = q(v(A))$;

   (ii) $q(v(\neg A)) = 1 - q(v(A))$;

   (iii) $q(v(1)) = 1$ and $q(v(\neg(A\vee\neg A)) = 0$;

   (iv) $q(v(Q(A\vee B))) = \max\{q(v(QA)) + q(v(QB)),1\}$.

If Γ is a non-empty set of wffs then we say that Γ *implies* A at x in M, M: $\Gamma \models_x A$ iff $\forall B\in\Gamma(v(B,x) \le v(A,x))$, Γ M-implies A, M: $\Gamma \models A$ iff either $\exists B\in\Gamma(x\notin\|B\|_{(-)})$, i.e. when B is not verified at x (verification but not truthfulness since we deal with many-valued logical matrix), or Γ implies A at all x in M. If we define $\Im = \langle X,\perp,\xi\rangle$ be QLO-MV-frame then Γ $\Im$-implies A iff $\Im$: $\Gamma \models A$ for all QLO-MV-models M on $\Im$. If $\wp$ is a class of QLO-MV-frames then Γ $\wp$-implies A, $\wp$: $\Im$:$\Gamma \models A$ iff $\Im$:

$\Gamma \models A$ for all $\Im \in \wp$. A class $\wp$ is said to *determine* QLO iff for all $A,B \in \Phi$, $A \Rightarrow B$ iff $\wp: A \models B$. $\wp$ *strongly determines* QLO iff for all Γ, A, $\Gamma \Rightarrow A$ iff $\wp: \Gamma \models A$.

If we define a range of values of a formula A as $\|A\| = \bigcup_{a \in \boldsymbol{R}} \|A\|_a$ then extending this definition on 4.1(1)-(7) hereafter we denote as $\|p_i\|$, $\|A \vee B\|$, $\|A \wedge B\|$, $\|\neg A\|$, $\|J_{|\alpha|}A\|$, $\|1\|$, $\|QA\|$ the ranges of respective formulas while $\|A\|_{(-)}$ means an arbitrary value of respective formula.

Lemma 4.2. *For any QLO-MV-model M and any $A \in \Phi$, $\|A\|_{(-)} \in \xi$.*

Proof. By induction on the length of A, exploiting 4.1. □

Theorem 4.3. (*Soundness of QLO-MV*). $\Gamma \Rightarrow A$ *if* Θ: $\Gamma \models A$, *where* Θ *is a class of all QLO-MV-frames.*

Proof. The proof, by induction on QLO-MV-derivability, proceeds by showing that the result holds for all Ax1-Ax22 and is preserved by application of Rx1-Rx6. We consider only the less obvious cases.

Ax2. Let $x \in \|A\|_a$. Then if $y \in \|\neg A\|_{-a}$, by 4.1(4) $y \perp x$ and hence (symmetry) $x \perp y$. 4.1(4) again gives $x \in \|\neg\neg A\|_a$.

Now let $x \in \|\neg\neg A\|_a$. Then $y \in \|\neg A\|_{-a}$ only if $x \perp y$, i.e. $y \perp \|A\|_a$ only if $x \perp y$. But $\|A\|_a$ is $\perp$-closed by 4.2 and thus $x \in \|A\|_a$.

Ax6. It is easy to make sure that $v(\neg(A \vee \neg A), x) = 0$ at any point $x \in X$ and likewise $v(A \vee \neg A, x) = 0$. But if $x \in \|\neg(A \vee \neg A)\|_0$, then $y \in \|A \vee \neg A\|_0$ just in case of $x \perp y$. By 4.1(2) $y \in \|A\|_b \cap \|\neg A\|_c$ and $b + c = 0$, i.e. $c = -b$. But then by 4.1(4) $y \perp y$ contrary to the irreflexivity of $\perp$. Hence, there is no y in any M for which we have $y \perp \|A \vee \neg A\|_0$, whence it follows by the definition that $x \in \|\neg(A \vee \neg A)\|_0$ for any x. Besides, for all B, by 4.1(3), $v(B \wedge B, x) \geq 0$.

Ax7. Let $x \in \|1 \wedge A\|_a$. Then $y \in \|1\|_1 \cap \|A\|_a$ by 4.1(3) and $v(1 \wedge A, x) = v(1,x) \circ v(A,x)$. But $v(1,x) = 1$ at any point $x \in X$ in virtue of the definition of (see 4.1(6)). So $v(1 \wedge A, x) = v(A,x)$ and thus M: $1 \wedge A \models A$ and M: $A \models 1 \wedge A$ for any A.

Ax11. Let $x \in \|J_\alpha(A \wedge B)\|_a$. Then by 4.1(5) $x \in \|A \wedge B\|_b$ and $a = \alpha b$. By 4.1(5) $x \in \|A\|_c \cap \|B\|_d$ and $b = cd$. Hence, $a = \alpha dc$. But by 4.1(5) $x \in \|J_\alpha A\|_{\alpha c} \cap \|B\|_d$ and by 5.3.3(3) $x \in \|J_\alpha A \wedge B\|_{\alpha cd = a}$.

Ax13. Let $x \in \|A\|_a$. Then by 4.1(5) $x \in \|J_\alpha A\|_{\alpha a}$ and by 4.1(4) $y \in \|\neg J_\alpha A\|_{-\alpha a}$ just in case of $x \perp y$. But then $y \in \|\neg A\|_{-a}$ because of $x \perp y$, and by 4.1(5) $y \in \|J_\alpha A\|_{-\alpha a}$.

Ax17. Let $x \in \|QQA\|_a$. Then by 4.1(7) $x \in \|QA\|_b$ and $a = q(b)$. Again, by 4.1(7) this implies $x \in \|A\|_c$ and $b = q(c)$. We have $q(b) = q(q(c)) = q(c)$ by the property of q and thus $a = q(c)$.

Ax18. Let $x \in \|A\|_b$. Then by 4.1(4) $y \in \|\neg A\|_{-b}$ only if $x \perp y$, i.e. $y \perp \|A\|_b$ only if $x \perp y$.Furthemore, by 4.1(7) $y \in \|Q\neg A\|_a$ and $a = 1 - q(b)$ according to the properties of q. But it is easy to check that the result will be the same for the right side of Ax18, i.e. $y \in \|1 \vee \neg QA\|_a$ and $a = 1\text{-}q(b)$.

Rx1. Suppose M: $A \models B$ and let $x\in\|\neg B\|_{-a}$. Then $y\in\|A\|_b$ only if $y\in\|B\|_a$ (by inductive hypothesis), only if $x\perp y$. This shows that $x\in\|\neg A\|_{-b}$.

The rest is obvious. □

Definition 4.4. Let L be a modal quantum logic of effects. The canonical QLO-MV-model of L is the structure $M_L = \langle X_L,\perp_L,\xi_L,v_L\rangle$, where:

(1) $X_L = \{x \subseteq \Phi$: x is a QLO-MV-full set$\}$;

(2) $x\perp_L y$ iff there is a wff A such that $Q\neg A\in x$, $A\in y$;

(3) $\xi_L = \{|A|^L: A\in\Phi\}$, where $|A|^L = \{x\in X_L: A\in x\}$;

(4) v_L: $(S\cup\Phi) \times X_L \to \mathbf{R}$.

Denoting $\{x\in X_L: v(A,x) = a\}$ as $\|A\|^L{}_a$ we come to the definition of the value of formula and ranges of valuation in canonical model M_L analogously to 4.1(1)-(7).

Lemma 4.5. $\Im_L = \langle X_L,\perp_L,\xi_L,v_L\rangle$ *is a QLO-MV-frame*.

Proof. Let $x\in X_L$. Then for any A neither $Q\neg A, A\in x$ nor x is QLO-MV-inconsistent (by Ax7). Hence, $x\perp_L x$ does not take place. If $x\perp_L y$, then for some wff A we have $Q\neg A\in x$, $A\in y$. By means of Ax2 we come to the conclusion that $Q\neg B\in y$, $B\in x$, where $B = Q\neg A$. Thus $x\perp_L y$ and $\perp_L$ is an orthogonality relation. To check whether $\|A\|^L$ be $\perp_L$-closed suppose that $x\notin\|A\|^L$, i.e. $A\notin x$. By Ax2 $\neg\neg A\notin x$ and so by 3.5 there is $y\in X_L$ such that $x\perp_L y$ fails and $\neg A\in y$. Meanwhile if $z\in\|A\|^L$ then $A\in z$ and, hence, $y\perp_L z$. Thus, $y\perp_L\|A\|^L$ as it was required. Clearly, ξ_L will be closed under intersection (by virtue of properties QLO-MV-derivability and QLO-MV-fullness). □

Theorem 4.6. (*Fundamental theorem for QLO-MV*). *For all A and all* $x\in X$, $x\in\|A\|^L$ *iff* $A\in x$.

Proof. By induction on the length of A. In case of $A = B\vee C$, $A=B\wedge C$, $A = J_\alpha B$ and $A = QB$ it is easy to see that $B,C\in x$ follows from $B\vee C$, $B\wedge C$, $J_\alpha B$, $QA\in x$. It will suffice to use 4.1(2),(3),(5),(7). Conversion follows from 3.2(3),(4),(5).

Suppose that $A = \neg B$ and for B the theorem is true. Let $Q\neg B\in x$. If $y\in\|B\|^L{}_{(-)}$ then by inductive hypothesis $B\in y$ and hence $x\perp_L y$. By 4.1(4) it follows that $x\in\|B\|^L{}_{(-)}$. Again, if $Q\neg B\notin x$, then according to 3.5 there is $y\in X_L$ such that $B\in y$ and thus by inductive hypothesis $y\in\|B\|^L{}_{(-)}$ but $x\perp_L y$ fails. By 4.1(4) we come to the conclusion that $x\notin\|B\|^L{}_{(-)}$. □

Corollary 4.7. $\Gamma\Rightarrow A$ *iff* M_L:$\Gamma \models A$.

Proof. If $\Gamma\Rightarrow A$ then there are $B_1,\ldots,B_n\in\Gamma$ such that either $B_1\vee\ldots\vee B_n\Rightarrow A$ or $(B_1\wedge\ldots\wedge B_n)\wedge(B_n\wedge\ldots\wedge B_1)\Rightarrow A$, or $J_{|\alpha_i|}B_i\Rightarrow A$, or $QB_i\Rightarrow A$

$(1 \le i \le n)$. If $x \in \|B\|^{L}_{(-)}$ for all $B \in \Gamma$, then by 4.6 $B_1,\ldots,B_n \in x$. By 4.1(2)-(5) it follows that $A \in x$ and thus $x \in \|A\|^{L}_{(-)}$.

The other way round, if A is not QLO-MV-derivable from Γ, then by 3.4 there exists $x \in X_L$ such that $\Gamma \subseteq x$ and $A \notin x$. Then by 4.6 $x \in \|B\|^{L}_{(-)}$ for all $B \in \Gamma$, but $x \notin \|A\|^{L}_{(-)}$. □

Theorem 4.8. $\Gamma \Rightarrow A$ *iff* $\mathfrak{I}_L : \Gamma \models A$.

Proof. Let $\boldsymbol{M}$ be an arbitrary QLO-MV-model on $\mathfrak{I}_L$. For every $i<\omega$, $\|p_i\|^{\boldsymbol{M}} \in \xi_L$ there is B_i such that $\|p_i\|^{\boldsymbol{M}}_{(-)} = |B_i|^{L}_{(-)}$ ($|B_i|^{L}_{(-)}$ is defined as in 4.4) and $|B_i|^{L}_{(-)} = \|B_i\|^{L}_{(-)}$. For any wff C let C' is the result of uniformly replacing each p_i, occurring in C, with B_i. Clearly, there are in Γ such $A_1,\ldots,A_n$ that either $A_1 \vee \ldots \vee A_n \Rightarrow A$ or $(A_1 \wedge \ldots \wedge A_n) \wedge (A_n \wedge \ldots \wedge A_1) \Rightarrow A$, or $J_{|\alpha_i|} A_i \Rightarrow A$, or $QB_i \Rightarrow A$ $(1 \le i \le n)$ and so we have $A'_1 \vee \ldots \vee A'_n \Rightarrow A'$ etc. Then by 4.7 either $\boldsymbol{M}_L$: $A'_1 \vee \ldots \vee A'_n \models A'$, or $\boldsymbol{M}$: $(A'_1 \wedge \ldots \wedge A'_n) \wedge (A'_n \wedge \ldots \wedge A'_1) \models A'$, or $\boldsymbol{M}_L$: $J_{|\alpha i|} A'_i \models A'$, or $\boldsymbol{M}_L$: $QA'_i \models A'$. But a simple induction shows that $\|C\|^{\boldsymbol{M}} = \|C'\|^{\boldsymbol{M}_L}$ and so either $\boldsymbol{M}_L$: $A_1 \vee \ldots \vee A_n \models A$, or $\boldsymbol{M}_L$: $(A_1 \wedge \ldots \wedge A_n) \wedge (A_n \wedge \ldots \wedge A_1) \models A$, or $\boldsymbol{M}_L$: $J_{|\alpha_i|} A_i \models A$, or $\boldsymbol{M}_L$: $QA_i \models A$ whence it follows that $\boldsymbol{M}$: $\Gamma \models A$. Since this holds for all models $\boldsymbol{M}$ on $\mathfrak{I}_L$, we conclude $\mathfrak{I}_L$: $\Gamma \models A$. □

Corollary 4.9. (*Strong completeness for QLO-MV*). *Θ: Γ ⊨ A only if Γ ⇒ A.*

Proof. Since by 4.5 $\mathfrak{I}_L$ is QLO-MV-frame, then Θ contains $\mathfrak{I}_L$ as its element. The rest is obvious. □

Thus corollary 4.9 shows that QLO-MV is strongly determined by the class of all QLO-MV-frames.

5. Bimodal Quantum Logic of Effects

Regarding effects of a Hilbert space as bounded linear operators E such that for all density operators D, $0 \le \mathrm{Tr}(DE) \le 1$, we can define over the class $\boldsymbol{E}(\boldsymbol{H})$ of all effects a partial ordering relation ≤ in the following way [2, p.397]. For any $E,H \in \boldsymbol{E}(\boldsymbol{H})$:

$E \le H$ iff for all density operators D: $\mathrm{Tr}(DE) \le \mathrm{Tr}(DF)$.

The class of all effects coincides with the class of all bounded linear operators between *0* and *1*. Clearly, $\boldsymbol{E}(\boldsymbol{H})$ contains the class of all $\lambda 1$ (with $\lambda \in [0,1]$) where for any state vector $\varphi \in \boldsymbol{H}$ $(\lambda 1)\varphi := \lambda\varphi$. Now we define for any $E,H \in \boldsymbol{E}(\boldsymbol{H})$:

$$E \oplus F := \begin{cases} E + F & \text{if } E + F \in \boldsymbol{E}(\boldsymbol{H}) \\ 1, & \text{otherwise} \end{cases}$$

where + the usual operator-sum,

$E^* := 1 - E$.

It is easy to see that

$E \oplus F = E + F$ iff $E + F \leq 1$.

Likewise one can easily check that the structure $\boldsymbol{E}(\boldsymbol{H}) = \langle \boldsymbol{E}(\boldsymbol{H}), \oplus, *, 1, 0 \rangle$ violates Łukasiewicz axiom of MV-algebra. Actually, let us consider two non-trivial effects E,F such that it's not the case that $E \leq F$ and it's not the case that $F \leq E$. Then, by definition of $\oplus$ we have $E \oplus F^* = 1$ and $F \oplus E^* = 1$. Hence, $(E^* \oplus F)^* \oplus F = 0 \oplus F = F \neq E = 0 \oplus E = (E \oplus F^*)^* \oplus E$. Thus, Łukasiewicz axiom is violated in the structure $\boldsymbol{E}(\boldsymbol{H})$.

As it was mentioned above R.Giuntini [2] showed that the class of all effects (determines by Born probability) of any Hilbert space turns out to be an instance of an algebraic structure called *quantum MV algebra* (QMV algebra). The latter is a structure $\boldsymbol{M} = \langle M, \oplus, *, 1, 0 \rangle$ where M is non-empty set, 0 and 1 are constant elements of M, $\oplus$ is a binary operation and * is a unary operation satisfying the following axioms (where $a \otimes b := (a^* \oplus b^*)^*$, $a \sqcap b := (a \oplus b^*) \otimes b$ and $a \sqcup b := (a \otimes b^*) \oplus b$):

(QMV1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

(QMV2) $a \oplus 0 = a$

(QMV3) $a \oplus b = b \oplus a$

(QMV4) $a \oplus 1 = 1$

(QMV5) $(a^*)^* = a$

(QMV6) $0^* = 1$

(QMV7) $a \oplus a^* = 1$

(QMV8) $a \sqcup (b \sqcap a) = a$

(QMV9) $(a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c)$

(QMV10) $a \oplus (b \sqcap (a \oplus c)^*) = (a \oplus b) \sqcap (a \oplus (a \oplus c)^*)$

(QMV11) $a \oplus (a^* \sqcap b) = a \oplus b$

(QMV12) $a \oplus (a^* \oplus b) \sqcup (b^* \oplus a) = 1$

It seems possible to yield logic of effects in QLO framework corresponding quantum MV algebra. To this end we will enrich the language of QLO with the help of a binary modal operator $\oplus$ and unary modal operator * and enlarge the list of QLO axiom with the following inference rules:

Rx7. $$\frac{\neg(C \vee \neg C) \Rightarrow A \Rightarrow 1 \quad \neg(C \vee \neg C) \Rightarrow B \Rightarrow 1 \quad A \vee B \Rightarrow 1}{A \oplus B \Leftrightarrow A \vee B}$$

Rx8. $$\frac{1 \Rightarrow A \oplus B}{1 \Leftrightarrow A \oplus B}$$

Rx9. $$\frac{\neg(A \vee \neg A) \Rightarrow A \Rightarrow 1}{1 \vee \neg A \Leftrightarrow A^*}$$

(the double line means an inference in both directions).

Let us denote a system QLO + {Rx7-Rx9} as QLO-QMV (with Ax1′). As in QLO we define $[A] \oplus [B] = [A \oplus B]$ and $[A]^* = [A^*]$.

Theorem 5.1. *A structure* $\boldsymbol{F} = \langle F, \oplus, ^*, 0, 1\rangle$ *where* $F = \{P/_{\sim}: P$ *is a formula and* $\neg(A \vee \neg A) \Rightarrow P \Rightarrow 1\}$, $0 = [\neg(A \vee \neg A)]$, $1 = [1]$, *представляет собой QMV алгебру*.

Proof. It is easy to see that satisfiability of (*QMV*1) and (*QMV*3) is a consequence of associativity and commutativity of $\vee$. (*QMV*2) will take place in virtue of Bx2. We have (*QMV*4) because from $\neg(A \vee \neg A) \Rightarrow B$ (by Rx7) one get $\neg(A \vee \neg A) \oplus 1 \Rightarrow B \oplus 1$ (by Rx4), and since by Bx1 $\neg(A \vee \neg A) \oplus 1 \Leftrightarrow 1$ then by Rx8 $1 \Leftrightarrow B \oplus 1$. In case of (*QMV*5) by Rx11 we have $1 \vee \neg A \Leftrightarrow A^*$, then again implementing Rx9 we obtain $1 \vee \neg(1 \vee \neg A) \Leftrightarrow A^{**}$. But by Bx2 this reduces to $1 \vee \neg 1 \vee \neg\neg A) \Leftrightarrow A^{**}$, which in view of $1 \vee \neg 1 \Leftrightarrow \neg\neg 1 \vee \neg\neg\neg 1 \Leftrightarrow \neg(\neg 1 \vee \neg\neg 1) \Leftrightarrow \neg(1 \vee \neg 1)$ (by Ax2, Ax1') and Ax2, Ax1' reduces, in turn, to $A \Leftrightarrow A^{**}$. Analogous manipulations allow to ascertain the satisfiability of (*QMV*6) and (*QMV*7).

In order to check the satisfiability of the remainder axioms we define $A \otimes B \Leftrightarrow (A^* \oplus B^*)^* \Leftrightarrow A \vee B \vee \neg 1$,

$A \otimes B \Leftrightarrow (A^* \oplus B^*)^*$,

$A \sqcap B \Leftrightarrow (A \oplus B^*) \otimes B$ and $A \sqcup B \Leftrightarrow (A \otimes B^*) \oplus B$.

Moreover, we obtain that

$$A \sqcap B \Leftrightarrow \begin{cases} A, & \text{if } A \Rightarrow B \\ B, & \text{otherwise} \end{cases} \qquad A \sqcup B \Leftrightarrow \begin{cases} A, & \text{if } B \Rightarrow A \\ B, & \text{otherwise} \end{cases}$$

Actually, by the definition $A \sqcap B \Leftrightarrow (A \oplus B^*) \otimes B \Leftrightarrow (A \oplus B^*) \vee B \vee \neg 1$. If $A \Rightarrow B$ then $A \oplus B^* \Rightarrow B \oplus B^* \Leftrightarrow 1$ and by virtue of Rx7 and Rx9 $A \oplus B^* \Leftrightarrow A \vee \neg B \vee 1$, and thus $A \sqcap B \Leftrightarrow A$. Otherwise by Rx11 $A \oplus B^* \Leftrightarrow 1$ and $A \sqcap B \Leftrightarrow B$.

Further, by the definition we have $A \sqcup B \Leftrightarrow (A \otimes B^*) \oplus B$. If $B \Rightarrow A$, then $B \otimes B^* \Rightarrow A \otimes B^*$, which leads to $\neg(B \vee \neg B) \Rightarrow A \otimes B^*$. This gives us an opportunity to exploit Rx7 for calculating $(A \otimes B^*) \oplus B$, which gives $(A \otimes B^*) \oplus B \Leftrightarrow (A \otimes B^*) \vee B \Leftrightarrow A \vee B^* \vee B \vee \neg 1 \Leftrightarrow A \vee 1 \vee \neg B \vee B \vee \neg 1 \Leftrightarrow A$. Otherwise we get $A \otimes B^* \Rightarrow \neg(B \vee \neg B)$. But by Rx7 we obtain that from $A \oplus B \Rightarrow \neg(B \vee \neg B)$ it follows $A \oplus B \Leftrightarrow \neg(B \vee \neg B)$, and thus $A \otimes B^* \Leftrightarrow \neg(B \vee \neg B)$ and $(A \otimes B^*) \oplus B \Leftrightarrow B$.

In case of (*QMV8*) if $A \Rightarrow B$ then $A \sqcup (B \sqcap A) \Leftrightarrow A \sqcup B \Leftrightarrow A$. If it is not the case that $A \Rightarrow B$, then $A \sqcup (B \sqcap A) \Leftrightarrow A \sqcup A \Leftrightarrow A$.

For (*QMV9*) we need that $(A \sqcap B) \sqcap C \Leftrightarrow (A \sqcap B) \sqcap (B \sqcap C)$. Two cases are possible:

1) $B \Rightarrow C$,

2) it is not the case that $B \Rightarrow C$.

Case 1). If $A \Rightarrow B$ then by virtue Rx3 $A \Rightarrow C$. Then $(A \sqcap B) \sqcap C \Leftrightarrow A \sqcap C \Leftrightarrow A \Leftrightarrow A \sqcap B \Leftrightarrow (A \sqcap B) \sqcap (B \sqcap C)$. If it is not the case that $A \Rightarrow B$, then $(A \sqcap B) \sqcap (B \sqcap C) \Leftrightarrow B \sqcap B \Leftrightarrow B \Leftrightarrow B \sqcap C \Leftrightarrow (A \sqcap B) \sqcap C$.

Case 2). Since it is not the case that $B \Rightarrow C$, then we get $B \sqcap C \Leftrightarrow C$. Hence, $(A \sqcap B) \sqcap (B \sqcap C) \Leftrightarrow (A \sqcap B) \sqcap C$.

In order that (*QMV10*) is satisfied we need $A \oplus (B \sqcap (A \oplus C)^*) \Leftrightarrow (A \oplus B) \sqcap (A \oplus (A \oplus C)^*)$. Two cases are possible:

1) $A \oplus C \Leftrightarrow 1$,

2) it is not the case that $A \oplus C \Leftrightarrow 1$.

Case 1). $A \oplus (B \sqcap (A \oplus C)^*) \Leftrightarrow A \oplus (B \sqcap \neg(A \vee \neg A)) \Leftrightarrow A$ and $(A \oplus B) \sqcap (A \oplus (A \oplus C)^*) \Leftrightarrow (A \oplus B) \sqcap (A \oplus \neg(A \vee \neg A)) \Leftrightarrow ((A \oplus B) \oplus A^*) \otimes A \Leftrightarrow ((B \oplus (A \oplus B^*)) \otimes A \Leftrightarrow (B \oplus 1)) \otimes A \Leftrightarrow A$.

Case 2) has two subcases:

a) $B \Rightarrow (A \oplus C)^*$,

b) it is not the case that $B \Rightarrow (A \oplus C)^*$.

Subcase a). By hypothesis, $A \oplus (B \sqcap (A \oplus C)^*) \Leftrightarrow A \oplus B$ and $(A \oplus B) \sqcap (A \oplus (A \oplus C)^*) \Leftrightarrow (A \oplus B) \sqcap (A \oplus (A \vee C)^*)$. If $(A \oplus (A \vee C)^*) \Leftrightarrow 1$ then we succeed. Therefore we can suppose that $(A \oplus (A \vee C)^*) \Leftrightarrow 1$ is not the case. Then $(A \oplus (A \vee C)^*) \Leftrightarrow (A \vee (A \vee C)^*) \Leftrightarrow C^*$. Thus $(A \oplus B) \sqcap (A \oplus (A \vee C)^*) \Leftrightarrow (A \oplus B) \sqcap C^*$. By hypothesis, $B \Rightarrow (A \oplus C)^* \Leftrightarrow 1 \vee \neg A \vee \neg C$, hence, $A \vee B \Rightarrow C^*$. Finally, $(A \oplus B) \sqcap C^* \Leftrightarrow A \oplus B$.

Subcase b). By hypothesis, we have that $A \oplus (B \sqcap (A \oplus C)^*) \Leftrightarrow A \oplus (B \sqcap (A \vee C)^*) \Leftrightarrow A \oplus (A \vee C)^* \Leftrightarrow A \vee (A \vee C)^* \Leftrightarrow C^*$. Now, $(A \oplus B) \sqcap (A \oplus (A \oplus C)^*) \Leftrightarrow (A \oplus B) \sqcap C^*$. By hypothesis, it is not the case that $B \Rightarrow (A \oplus C)^*$. Then it is not the case that $C \Rightarrow (A \oplus B)^*$, hence it is not the case that $(A \oplus B) \Rightarrow C^*$. Thus, $(A \oplus B) \sqcap C^* \Leftrightarrow C^*$.

Cases of (*QMV11*) and (*QMV12*) are easily verified. □

In the sequel under wff we always mean wff P, for which $\neg(A \vee \neg A) \Rightarrow P \Rightarrow 1$ is true.

Definition 5.2. Let Γ be a non-empty set of wffs. A wff A is said to be QLO-QMV-*derivable* from Γ, $\Gamma \Rightarrow A$, if A is QLO-*derivable* from Γ and there exist $B_1, \ldots, B_n \in \Gamma$, such that

(a) $B_1 \oplus \ldots \oplus B_n \Rightarrow A$.

The notions of QLO-QMV-derivability, QLO-QMV-consistency etc. are defined in the same way as in case of QLO-MV (it can be shown that Γ is QLO-QMV-consistent iff for no A do we have both $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow A^*$). Γ is QLO-QMV-*full* iff it is QLO-full and $A, B \in \Gamma$ implies $A \oplus B \in \Gamma$.

Lemma 5.3. *If $x \subseteq \Phi$ (where Φ is a set of wff) is QLO-QMV-full, then*

(1) *$x \Rightarrow A$ iff $A \in x$;*

(2) *$1 \in x$ for all wff A.*

Proof. (ii) By definition x is non-empty, thus there exists $B\in x$. But by 5.2 $J_0B\in x$, so by Ax9 and 3.2(2) we have $\neg(A\vee\neg A)\in x$. But since for wff P always will be true that $\neg(A\vee\neg A) \Rightarrow P \Rightarrow 1$, then by 5.2 we obtain the desired result. □

QLO-QMV-full sets and QLO-QMV-derivability are connected with the following version of Lindenbaum's Lemma :

Theorem 5.4. *$\Gamma\Rightarrow A$ iff A belongs to every QLO-QMV-full extension of* Γ.

Proof. We need verify only the cases of $B_1\oplus\ldots\oplus B_n\Rightarrow A$ and $B\oplus C\in x$. □

Theorem 5.5. *If x is QLO-QMV-full and $A^*\notin x$, then there exists QLO-QMV-full set y such that $A\in y$ and for all B, either $B^*\notin x$ or $B\notin y$.*

Proof. Let $y = \{B\text{: } A\Rightarrow B\}$. By Ax1 $A\in y$. Now let $B^*\in x$, that implies $\neg B\in x$ by Rx9. Then $B\notin y$, or else $A\Rightarrow B$ and $\neg B \Rightarrow \neg A$ by Rx1, $\neg A\in x$ and then $A^*\in x$ contrary to hypothesis. Further, by 5.3 we have $1\in x$. According to what we just proved, by Rx9 we obtain that $1\vee\neg 1\notin y$. Proceeding in a similar manner to 5.4 we can show that y is closed under QLO- and QLO-QMV-derivability. Then since $1\vee\neg 1$ is not QLO-QMV-derivable from y, i.e. y is QLO-QMV-consistent, y be QLO-QMV-full as required. □

6. Semantics of QLO-QMV

Since our system is an extension of QLO then for its description we will exploit the definitions of QLO-semantics modifying them as required to convey specificity of QLO-QMV.

Definition 6.1. QLO-QMV-models are QLO-models enriched with the following two points in recursive definition of the value of wff:

(1) $\|A\oplus B\|_a = \{x\in X\text{: } x\in\|A\|_b\cap\|B\|_c \;\&\; a = \min(1,b + c)\}$;

(2) $\|A^*\|_a = \{x\in X\text{: } x\perp\|A\|_{1-a} \;\&\; v(A^*,x) = 1\text{-}a\}$.

Lemma 6.2. *For any QLO-QMV-model **M** and any $A\in\Phi$, $\|A\|_{(\text{-})}\in\xi$.*

Proof. By induction on the length of A, exploiting 6.1. □

Theorem 6.3. (*Soundness of QLO-MV*). *$\Gamma\Rightarrow A$ if Θ: $\Gamma\models A$, where Θ is a class of all QLO-QMV-frames.*

Proof. The proof, by induction on QLO-QMV-derivability, proceeds by showing that the result holds for all Ax1-Ax16 and is preserved by application of Rx1-Rx5, Rx7-Rx9. We consider only the cases of Rx7-Rx9 (accounting of the proof for QLO).

Rx7. By hypothesis, ***M***:$\neg(C\vee\neg C)\models A$, ***M***:$A\models\mathbf{1}$, ***M***:$\neg(C\vee\neg C)\models B$, ***M***:$B\models 1$, ***M***:$A\vee B\models 1$. According to the definition ***M***:$\neg(C\vee\neg C)\models_x A$ iff $v(\neg(C\vee\neg C),x) \leq v(A,x)$, i.e. $0 \leq v(A,x)$, and ***M***:$A\models_x 1$ iff $v(A,x) \leq v(1,x)$, i.e. $v(A,x)\leq 1$. The same is true for B. Besides, we have $v(A\vee B,x) \leq v(1,x)$. Then by 4.1(2), 6.2(1) we obtain $(A\vee B,x) = v(A,x) + v(B,x) = v(A\oplus B,x)$. In the reverse direction the proof is obvious.

Rx8. By hypothesis, $\boldsymbol{M}{:}1 \models A\oplus B$. Then we have $\boldsymbol{M}{:}1 \models_x A\oplus B$ iff $1\leq v(A\oplus B,x)$. But since $v(A\oplus B,x) = \min(1,v(A,x) + v(B,x))$, then, clearly, $1 = v(A\oplus B,x)$.

Rx9. Let $\boldsymbol{M}{:}\neg(C\vee\neg C) \models A$, $\boldsymbol{M}{:}A \models 1$. Then $\boldsymbol{M}{:}\neg(C\vee\neg C) \models_x A$ iff $0\leq v(A,x)$, and $\boldsymbol{M}{:}A \models_x 1$ iff $a = v(A,x) \leq 1$. Further, $y\in\|\neg A\|_{-a}$ only if $x\perp y$. By 4.1(2) we get $y\in\|1\vee\neg A\|_c$ iff $y\in\|1\|_1\cap\|\neg A\|_{-a}$ and $c = 1 - a$. But by 6.2(2) we have $y\in\|A^*\|_c$. □

Definition 6.4. The canonical QLO-QMV-model of L (bimodal quantum logic of effects) is the structure $\boldsymbol{M}_L = \langle X_L,\perp_L,\xi_L,v_L\rangle$ where:

(1) $X_L = \{x \subseteq \Phi\text{: } x \text{ is a QLO-QMV-full set}\}$;

(2) $x\perp_L y$ iff there is a wff A such that $A^*\in x$, $A\in y$;

(3) $\xi_L = \{|A|^L\text{: } A\in\Phi\}$, where $|A|^L = \{x\in X_L\text{: } A\in x\}$;

(4) $v_L\text{: } (S\cup\Phi) \times X_L \to \mathbf{R}$.

Lemma 6.5. $\Im_L = \langle X_L,\perp_L,\xi_L,v_L\rangle$ *is a QLO-QMV-frame.*

Proof. Argumentation is similar to the case of QLO-MV-frame. □

Theorem 6.6. (*Fundamental theorem for QLO-QMV*). *For all A and all* $x\in X$, $x\in\|A\|^L$ *iff* $A\in x$.

Proof. By induction on the length of A. In case of $A = B\oplus C$ it is easy to see that $B,C\in x$ follows from $B\oplus C\in x$. Suffice to use 6.1(1). □

***Corollary* 6.7.** $\Gamma\Rightarrow A$ *iff* $\boldsymbol{M}_L{:}\Gamma \models A$.

Proof. If $\Gamma\Rightarrow A$ then we need to consider an additional case of $B_1\oplus\ldots\oplus B_n\Rightarrow A$. If $x\in\|B\|^L_{(-)}$ for all $B\in\Gamma$, then by 6.6 $B_1,\ldots,B_n\in x$. By 6.1(1) it follows that $A\in x$ and thus $x\in\|A\|^L_{(-)}$.

The other way round, if A is not QLO-QMV-derivable from Γ, then by 5.4 there exists $x\in X_L$ such that $\Gamma\subseteq x$ and $A\notin x$. Then by 6.6 $x\in\|B\|^L_{(-)}$ for all $B\in\Gamma$, but $x\notin\|A\|^L_{(-)}$. □

Theorem 6.8. $\Gamma\Rightarrow A$ *iff* $\Im_L{:}\Gamma \models A$.

Proof. We need to consider an additional case of $A_1\oplus\ldots\oplus A_n\Rightarrow A$ for Γ. □

***Corollary* 6.9. (*Strong completeness for QLO-QMV*).** Θ: $\Gamma \models A$ *only if* $\Gamma\Rightarrow A$.

Proof. Since by 6.5 $\Im_L$ is QLO-QMV-frame, then Θ contains $\Im_L$ as its element. The rest is obvious. □

Thus corollary 6.9 shows that QLO-MV is strongly determined by the class of all QLO-MV-frames.

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