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S5 IS A PARACONSISTENT LOGIC AND SO IS FIRST-ORDER CLASSICAL LOGIC*

Abstract. *We present and discuss the fact that the well-known modal logic S5 and classical first-order logic are paraconsistent logics.*

> Quoi? quand je dis “Nicole, apportez-moi mes pantoufles,
> et me donnez mon bonnet de nuit”, c’est de la prose?
> Par ma foi! Il y a plus de quarante ans que je dis de la prose sans que j’en
> susse rien, et je vous suisle plus obligé du monde de m’avoir appris cela.
> Molière, *Le bourgeois gentilhomme*

1. Introduction

A *paraconsistent negation* is a unary operator ~ such that

(N) *a, ~ a* ® *b*

(P) the operator ~ has enough properties to be called a negation.

A *paraconsistent logic* is a logic with a paraconsistent negation.

The second property above is quite fuzzy, anyway there are in the litterature a bunch of operators that people agree to call paraconsistent negations, and consequently a bunch of logics which are called paraconsistent logics.

In this paper we will show that it is possible to define in *S5* (and in other logics such as classical first-order logic) a negation which can reasonably be considered as paraconsistent.

It seems that this simple fact has not yet been noticed, although there are people making investigations in paraconsistent logic since more than 30 years. For literature about paraconsistent logic the reader may consult [23. [24. [2.. He will see that such fact is not mentioned. The aim of this paper is to show that this fact is highly relevant and can be a new starting point for research in paraconsistent logic as well as in modal logic.

2. Basic properties of the paraconsistent negation of *S5*

Consider the standard language of *S5* with ¬, ◊, □, →, ∨, ∧. We define the operator ~ as follows:

$$\sim a =_{Def} \Diamond\neg a$$

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As in *S5* we have (about *S5* the reader may consult classical texts such as [19., [11., [10.):

$$a, \Diamond\neg a \circledR b$$

therefore ~ obeys the property (N) above.

Regarding the property (P), it is easy to check that the following are theorems of *S5*:

$$a \vee \sim a$$

$$(a \rightarrow \sim a) \rightarrow \sim a$$

$$(\sim a \rightarrow a) \rightarrow a$$

And we have the following theorems but not their converses:

$$(a \rightarrow b) \rightarrow (\sim a \vee b)$$

$$\sim \sim a \rightarrow a$$

$$\sim (a \wedge b) \rightarrow (\sim a \vee \sim b)$$

$$\sim (\sim a \wedge \sim b) \rightarrow (a \vee b)$$

$$\sim (a \wedge \sim b) \rightarrow (\sim a \vee b)$$

$$\sim (\sim a \wedge b) \rightarrow (a \vee \sim b)$$

$$\sim (\sim a \vee b) \rightarrow (a \wedge \sim b)$$

$$\sim (a \vee b) \rightarrow (\sim a \wedge \sim b)$$

$$\sim (a \vee \sim b) \rightarrow (\sim a \wedge b)$$

$$\sim (\sim a \vee \sim b) \rightarrow (a \wedge b)$$

As a matter of comparison, the four last are not theorems of da Costa's well-known paraconsistent logic *C1* (about *C1* see [12. [3.).

Furthermore, we have:

$$\sim (a \wedge \sim a)$$

This may seem strange, because sometimes this formula is considered as a formulation of the principle of non contradiction and sometimes paraconsistent logic is roughly speaking characterized as a logic in which this principle does not hold. However there are various paraconsistent logics studied in the litterature in which this formula holds. This is the case for example of paraconsistent logics defined with three-valued matrices where the third value 1/2 is taken as distinguished, like D'Ottaviano-da Costa's logic *J3* and Priest's logic *LP* (see [16., [22.). In these logics the negation of 1/2 is1/2 and the conjunction of 1/2 and 1/2 is1/2. It is easy to see then that the value of $\sim (a \wedge \sim a)$ is always distinguished. For paraconsistentists like Priest the value 1/2 is interpreted as true-false. Therefore this is an example of an intuitive interpretation under which the above formula is a paraconsistent tautology.

The definition of paraconsistent negation has been improved by Urbas [27. by substituing the property (NN) below for the property (N).

(NN) $a, \sim a$ ® b, for any schema b which is not tautological.

Urbas's definition (*strict paraconsistency*) permits to exclude out of the sphere of paraconsistency logics like Johansson's minimal logic where (N) holds but in which we have:

$a, \sim a$ ® b

As we can see the paraconsistent negation of *S5* is a strict paraconsistent negation.

Another good feature is that the bi-implication ($\leftrightarrow$) defined in the usual way is a congruence relation in S5, in particular we have:

$$\text{if } |\!- a \leftrightarrow b \text{ then } |\!- \sim a \leftrightarrow \sim b$$

This is not the case of the bi-implication of da Costa's logic *C1*, logic in which it is in fact not possible to define a non trivial congruence relation, as proved by Mortensen [20..

Like *C1* the logic *S5* has two negations, a classical one ($\neg$) and a paraconsistent one (~), and in *S5* it is possible to define, like in *C1*, the classical negation with the help of the paraconsistent one (and other connectives).

3. Classical first-order logic is paraconsistent

According to a theorem of Wajsberg (see [29.), it is possible to translate *S5* into the fragment of monadic classical first-order logic with only one variable and vice versa. Following the idea of this translation, we can define a paraconsistent negation into this logic like this:

$$\sim \varphi =_{Def} \exists x \neg \varphi$$

Due to Wajsberg's theorem, this negation has exactly the same properties as the one presented in the preceding section.

It was difficult to construct the first paraconsistent logics. Some people, like Popper, argued that it would not be possible to build a paraconsistent negation (see [21., [9., [26.). Various techniques more or less artificial were used. So it is an astonishing fact that a paraconsistent negation, and rather a good one, is already built in the most famous and recognized logic, classical first-order logic.

In view of this fact, one can argue that paraconsistent logic is not a deviant logic, an abnormal and monstrous creature threatening the very basis of rationality, democracy and monotheism. If paraconsistent logic is such a monster then it is rooted in what is considered as the core of rationality which is therefore deeply rotten and has to be clean up. Maybe one has to consider another first-order logic. But if we take for example intuitionistic first-order logic, it is easy to see that the same definition in monadic intuitionistic first-order logic with one variable leads also to a paraconsistent negation.

In the same way that Mr. Jourdain of Moliere's *Bourgeois gentilhomme* was making prose without knowing it, we can say that Mr. Frege and his successors were doing paraconsistent logic without knowing it. And if one argues that the founder of first-order logic is Frege or Peirce, one could argue therefore that Frege or Peirce is the real founder of paraconsistent logic. Or even Aristotle, if one considers that monadic first-order logic with one variable is already contained within syllogistic. This kind of strange considerations are just to show that it is difficult to argue that the creators of paraconsistent logic were people who developed logics containing implicitly a paraconsistent negation. The real creators of paraconsistent logic are people, like Jaskowski and da Costa, who were trying to construct explicit paraconsistent negations. Of course they could have realized that a paraconsistent negation was already at hand inside classical first-order logic, instead of building other negations in more or less artificial ways. (About the history of paraconsistent logic, see for example [15.).

4. Extracting paraconsistent logics from modal logics

Consider the function * from the set of formulas ***G*** built with ~, ∨, ∧, → into the set of formulas ***F*** built with ¬, ∨, ∧, →, ◊, □ defined by:

$a^* = a$, if a is atomic

$(a \oplus b)^* = a^* \oplus b^*$, where ⊕ is ∨, ∧ or →

$(\sim a)^* = \Diamond\neg(a)^*$

We call *PS5* the logic <***G***; ~; ∨; ∧; →; $\vdash_{PS5}$> such that:

$$T \vdash_{PS5} a \text{ iff } T^* \vdash_{S5} a^*$$

The decidability of *PS5* is a direct consequence of the decidability of *S5*.

It easy to define a semantics for this logic. Given a Kripke structure K with a universal relation of accessibility, we define the standard connectives as usual and the paraconsistent negation with the following condition:

~ *a* is false in the world W iff *a* is true in every world of K

A more difficult problem is how to axiomatize *PS5* (in a non trivial way). We have solve this problem in [5., presenting a sound and complete Hilbert-type system for *PS5*.

We can generalize the above idea and given a modal logic

$$M = \langle \boldsymbol{F}; \neg; \Diamond; \Box; \vee; \wedge; \rightarrow; \vdash_M \rangle$$

we can define the paraconsistent logic PM associated to it as the logic

$$PM = \langle \boldsymbol{G}; \sim; \vee; \wedge; \rightarrow; \vdash_{PM} \rangle$$

such that

$$T \mid\!-_{PM}\ a \text{ iff } T^* \mid\!-_M\ a^*$$

If *M* is decidable of course *PM* will be decidable, but it is not clear that the axiomatizability of *M* entails the axiomatizability of *PM*. Another point is that if one can reasonably expect, due to the basic properties of modalities, the negation ~ of *PM* to be a *paraconsistent* negation in the sense that it obeys the condition (N), it is not clear whether it would fullfil the condition (P), i.e. if it could properly be called a paraconsistent *negation*.

In Kripke semantics for intuitionistic logic, instead of the above semantical condition for negation, we have:

~ *a* is true in the world W iff *a* is false in every world *V* accessible from *W*

where the accessibility relation is a quasi-ordering. Another difference is that the implication is also defined with the help of the accessibility relation. One may want to consider the dual of this semantics and see if it defines the same logic as the sequent calculus dual of intuitionistic logic *LDJ* [28. or the algebraic dual of it [25.. Anyway it is for sure a paraconsistent logic. (Such semantics may have some connections with the one given by Baaz in [1. for da Costa's logic C_ω which has an intuitionistic implication.)

If we now consider the same condition as the one for intuitionistic negation but with a universal relation of accessibility, we have:

~ *a* is true in the world *W* iff *a* is false in every world of *K*

This condition is *dual* to the condition for the paraconsistent negation in *PS5* and together with the standard conditions for other connectives generates a paracomplete logic dual to the paraconsistent logic *PS5*. The paracomplete negation defined with this condition corresponds in *S5* to *not possible* (¬◊) like in Godel's translation of intuitionistic logic into *S4* (see [18.).

5. Generating modal logics from paraconsistent logics

Considering the converse procedure of the preceding section, given a paraconsistent logic P we can define a modal logic *MP* associate to it, that is to say a modal logic where ◊¬ behaves as ~ in P, ¬ being the classical negation. For example one can consider *MC1*, *MJ3*, *MLP*, *MLDJ*, modal logics associated respectively to the paraconsistent logics *C1*, *J3*, *LP*, *LDJ*.

The question will be then to know in which sense the modal operators generated by this means correspond intuitively to possibility and necessity. In the case of *C1*, we will get a modal logic, which is not a classical modal logic (in the sense of [11.). For example we will have:

$$a \leftrightarrow (a \wedge a)$$

but not

$$\Diamond\neg a \leftrightarrow \Diamond\neg(a \wedge a)$$

It would be also interesting to consider what kind of modal logics are associated, according to our definition, to De Morgan's paraconsistent logics, i.e. logics where are valid the laws:

$$\sim\sim a \leftrightarrow a$$

$$\sim (a \wedge b) \leftrightarrow (\sim a \vee \sim b)$$

$$\sim (a \vee b) \leftrightarrow (\sim a \wedge \sim b)$$

etc.

6. Prospects

This interplay between modal and paraconsistent logics seems promising both from the technical and philosophical sides.

Technically speaking, modal logic and paraconsistent logics are closely tied: they are both the study of unary connectives which differ from affirmation (a) or classical negation ($\neg a$). Of course intuitively modalities such as possibility and necessity must have different properties than paraconsistent negations. But the modal logician is led for technical reasons related to the systematization of his work to consider other unary operators than possibility, necessity, impossibility and contingency. When one speaks of irreducibility of modalities, one speaks about unary connectives which are not interdefinable, including such connective as $\Diamond\sim$ which turns out to be a paraconsistent negation.

Modal logicians have developed techniques such as Kripke semantics which have applications going far beyond the study of traditional modalities. For example Kripke semanticscan be used to define intuitionistic *negation*. As it is known intuitionistic negation is not truth-functional in the sense that it cannot be characterized by a finite matrix. There are some paraconsistent negations which are defined by finite matrices. But it will be interesting to see how we can distinguish these truth-functional paraconsistent negations from those who are not. According to Dugundji's theorem, *S5* (and other modal logics like *S4*, etc.) cannot be characterized by a finite matrix (see [17.). This result can be applied to the paraconsistent negations of these logics.

On the other hand paraconsistent negations defined via modalities are algebraizable using the standard methods of algebraization of modal logic. And this is an interesting feature because paraconsistent logic has not yet been treated in a satisfactory way by algebraic methods.

Generating modal logics in which there placement theorem does not hold, from paraconsistent logics like C1, applying semantical methods such as the theory of valuation [13. [14., can also be interesting because it seems that the idea of intentional operator is not compatible with such theorem (see [4.).

From the *philosophical* point of view it seems that the modal approach to paraconsistent negation defining such negation as *possibly not* can be fruitful. It is an intuitive idea, which can be exemplified and justified in many ways. Of course one has to examine if this really makes sense and if there are not technical results which go against this intuitive interpretation of paraconsistent negation. But in general it seems that such definition fits well with the intuition. For example let us examine the interesting case of double negation.

In natural language double negation is often used to emphasize a sentence in such away as if it was stronger than simple affirmation, as in the following example:

It is not true that God does not exist.

In a logic like *S5* in which we have

$$\sim\sim a \rightarrow a$$

but not

$$a \rightarrow \sim\sim a$$

double (paraconsistent) negation is really stronger than simple affirmation. The reason why in *S5*, is that double (paraconsistent) negation means necessity, as we can see:

In *S5*, we have:

$$\Diamond\Box a \leftrightarrow \Box a$$

and considering that:

$$\Diamond \sim \Diamond \sim a \leftrightarrow \Diamond\Box a$$

we have:

$$\sim\sim a \leftrightarrow \Box a$$

Therefore the above double negated sentence means from the point of view of the paraconsistent negation of *S5*:

God necessarily exists.

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