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COMBINED CAUSAL LOGICS OF MINKOWSKI SPACETIME*

Abstract. *Getting started from ideas of N.A.Vasiliev and exploiting some conceptions of G.Frege, V.A.Smirnov introduced several combined calculi of sentences and events consisting of two parts: the abstract (external) logic depending on epistemological assumptions and the empirical (internal) logic depending on ontological ones. Early the author proposed to approach algebra of events as the discursive system of S. Jaśkowski* (*cf.* [*5*]). *One more interesting possibility would be an exploitation of an S4.2-modal algebra instead of an S5-modal algebra for discursive logic as an algebra of events. As it was shown by R.Goldblatt* [*4*] *in the Diodorean interpretation of modality where the operator of necessity* $\Box$ *is read as "it is now and always will be the case that" time would be modelled by the four-dimensional Minkowskian geometry that forms the basis of Einstein's special theory of relativity. In this case "event" y coming after event x just in case a signal can be seen from x to y at a speed at most that of the speed of light (i.e. y is in the causal future of x). Passing to the S4.2-modal algebra of the "histories" (subsets of events or causal paths) we thus obtain a combined calculus of sentences and histories. The same would be done in a more abstract way if we consider an algebra of the histories as a complete orthomodular lattice following to W.Cegla's approach* (*cf.* [*2*]). *For both formulations of combined causal logic of Minkowski spacetime a semantic of (event) bundles and semantic of possible worlds is built and some metamathematical results are obtained.*

1. Introduction

An idea of distinguishing two levels in logic goes back to N.A.Vasiliev. He distinguished two levels in logic assuming inconsistency on ontological level, but denying it on logical one. Proceeding from this idea and exploiting some conceptions of G.Frege, V.A.Smirnov introduced several combined calculi of sentences and events (cf. [9]) consisting of two parts: the abstract (external) logic depending on epistemological assumptions and the empirical (internal) logic depending on ontological ones. Both external and internal logics are subjected to change. The language of combined calculi usually includes two sorts of variables: event variables (terms) and propositional ones. If a and b are terms then $a \cup b$, $a \cap b$, $\sim a$ will be also terms (complex events) while θa, θb are the formulas along with the formulas $\theta a \vee \theta b$, $\theta a \wedge \theta b$, $\neg \theta a$. Clearly, postulating some equivalencies

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like $\theta(a\cup b) \equiv \theta a\vee\theta b$, $\theta(a\cap b) \equiv \theta a\wedge\theta b$ etc. we arrive at different combination of algebras of events and propositional calculi in the framework of one logic.

Early the author proposed to approach algebra of events as the discursive system of S. Jaśkowski (cf. [5]) by treating algebra of events as an $S5$-modal algebra, introducing Jaśkowski's type conditional and then θ-translating it into classical sentential calculus (cf. [10]). One more interesting possibility would be an exploitation of an $S4.2$-modal algebra instead of an $S5$-modal algebra as an algebra of events. As it was shown by R.Goldblatt [4] in the Diodorean interpretation of modality where the operator of necessity $\Box$ is read as "it is now and always will be the case that" time would be modelled by the four-dimensional Minkowskian geometry that forms the basis of Einstein's special theory of relativity. In this case "event" y coming *after* event x just in case a signal can be seen from x to y at a speed *at most* that of the speed of light (so that y is in the causal future of x). The modal sentences valid in this structure are precisely the theorems of the logic $S4.2$. Passing to the $S4.2$-modal algebra of the "histories" (subsets of events or causal paths) and θ-translating those into classical sentential calculus (in a way it was done in [10] in case of $S5$-algebra) we thus obtain a combined calculus of sentences and histories.

But this is not the end of a story since the same would be done in a more abstract way. Let (X,G) be a pair where X is a non-empty set and G is a structure defined by a distinguished covering G of X by non-empty subsets. The elements $f\in G$ will be called causal paths or "histories" and let us denote by $\beta(x) = \{f\in G: x\in f\}$ the set of all paths containing x. Two points x and y are *causally related* if there is some path f containing both of them.

In [2] W.Cegla shows that in a causal space (X,G) one can introduce an orthogonality relation in the following way: given $x,y\in X$, x is orthogonal to y iff $x \neq y$ and there is no $f\in G$ such that $x\in f$ and $y\in f$ (this means that x and y are not causally related). As a consequence we are able to introduce an orthogonal complement by means of the following definition: if $A\subset X$ then $A^{\perp} = \{x\in X: \forall f\in\beta(x)(f\cap A = \varnothing)\}$. If we take the family $l(X,\perp) = \{A\subset X: A = A^{\perp\perp}\}$ it is well-known that $l(X,\perp)$, partially ordered by set-theoretic inclusion with the orthocomplementation $A\rightarrow A^{\perp}$ and $\vee A_i = (\cup A_i)^{\perp\perp}$, $\wedge A_i = \cap A_i$, forms a complete ortholattice.

Moreover, we can introduce the second family of sets $L(X,\perp) = \{D^{\perp\perp}: D \text{ is an orthogonal set}\}$ where if $D\subset X$, then D is an *orthogonal set* iff $\forall x,y\in D\forall f\in\beta(x)(f\cap y = \varnothing)$. The family $L(X,\perp)$ will be an orthocomplete orthoposet. Indeed, $L(X,\perp) \subset l(X,\perp)$ but there is a case when they are identical. If D is orthogonal set of X, and $x\in X$, $x\notin D^{\perp}$,

$x \notin D^{\perp\perp}$ then $D^{\perp} \cap (x^{\perp\perp} \cap D^{\perp})^{\perp} \neq \varnothing$. In this case $l(X,\perp) = L(X,\perp)$ is a complete orthomodular lattice.

Both algebraic causal structures are connected with localization in non-relativistic and relativistic cases. These algebras of the "histories" (causal paths) also would be θ-translating into classical sentential calculus (in a way it was done in [10] in case of *S*5-algebra) and we thus again arrived at a combined calculus of sentences and histories.

In the paper a semantic of (event) bundles and semantic of possible worlds for both formulations of combined causal logic of Minkowski spacetime is proposed reflected the different treatment of the events and histories.

2. Diodorean Approach to Combined Causal Logic of Minkowski Spacetime

In a nutshell the main points of an approach to algebra of events as the discursive system of S. Jaśkowski (cf. [10]) would be posed as follows.

In his paper "Propositional Calculus for Contradictory Deductive Systems" S.Jaśkowski [5] offers a system of discursive logic by adding to S5 modal system a conditional $\rightarrow$ (often written as $\supset_d$ and called discursive implication) and defining $\alpha\rightarrow\beta$ as $\Diamond\alpha\rightarrow\beta$. The logical truths of the pure $\rightarrow$ fragment of discursive logic are the same as those of the pure $\supset$ fragment of classical logic but unlike of the latter $\models \alpha\rightarrow(\neg\alpha\rightarrow\beta)$ fails, since $\models_{S5} \Diamond(\Diamond\alpha\supset(\Diamond\neg\alpha\supset\beta))$ fails too.

Approaching algebra of events as S5-modal algebra one is in position to cope with contradictory character of ontological level by introducing counterpart of Jaśkowski's type conditional in algebra of events and then θ-translating it into our sentential calculus. The only difficulty appears to be the nature of the possible event (are there some criteria for dividing events into possible and real one?) This would be obviated by means of the "making possible" modal operator

$$\boldsymbol{MP}(x,y)\leftrightarrow y\in\sigma(x)$$

(*x makes possible y iff y is synthetizable from x*) [8]. The possible event ontologically means that we (i) purport possibility as the case when a relation between some event and possible event take place and (ii) identify with this relation the relation "making possible" (a kind of "maker"). Thus, in a sense, possible events are "ontologically generated" by some other events.

To explicate theorems of Jaśkowski's discursive logic in the system of combined logic one should add the following schemes to the axiom schemata of classical sentential logic and the rule *modus ponens*:

$\theta a \vee \theta b \equiv \theta(a \cup b)$

$\neg\theta a \equiv \theta(\sim a)$
$\theta(\Diamond(a \cup b)) \equiv \theta(\Diamond a) \vee \theta(\Diamond b)$
$\theta a \supset \theta(\Diamond a)$
$\theta(\Diamond\Diamond a) \supset \theta a$
$\theta(\Diamond a) \supset \theta(\sim\Diamond\sim\Diamond a)$

Let us hereafter $a \to b$ means $\sim\Diamond a \cup b$, $a \leftrightarrow b$ means $(\sim\Diamond a \cup b) \cap \cap (\sim\Diamond a \cup \Diamond b)$. It easily can be seen that the first two axioms provide us with a Boolean algebra structure of the set of events and the following theses will take place:

$\theta a \wedge \theta b \equiv \theta(a \cap b)$
$\theta(a \to b) \supset (\theta a \supset \theta b)$
$\theta(a \leftrightarrow b) \supset (\theta a \equiv \theta b)$

There is also an operator [-] in the language of this calculus JVCD of Jaskowśki-Vasiliev Combined Discursive Logics such that if α is a formula then $[\alpha]$ is a sentential term and thus one may relate with an every formula the respective event (“the event, that α“). By means of such an operator JVCD-system is enriched with the axioms

$\theta[\alpha] \equiv \alpha$
$\theta[\alpha \vee \beta] \equiv \theta([\alpha] \cup [\beta])$
$\theta[\neg\alpha] \equiv \theta(\sim[\alpha])$
$\alpha \supset \theta(\sim[\alpha] \to \mathrm{b})$

where *b* is an arbitrary event in the event algebra.

And now what should be changed if we need an S4.2-algebra of events instead of S5-algebra on ontological level? Technically it involves a passage to the following axioms:

A1. $\theta a \vee \theta b \equiv \theta(a \cup b)$
A2. $\neg\theta a \equiv \theta(\sim a)$
B1. $\theta(\Diamond(a \cup b)) \equiv \theta(\Diamond a) \vee \theta(\Diamond b)$
B2. $\theta a \supset \theta(\Diamond a)$
B3. $\theta(\Diamond\Diamond a) \supset \theta a$
B4. $\theta(\Diamond\sim\Diamond\sim a) \supset \theta(\sim\Diamond\sim\Diamond a)$

Here the operator $\Diamond$ means "it will (at some time) be" and thus a "synthetizability" correlates with the time ordering.

A semantic of such combined logic would be described as follows. Let a structure $\mathbf{T} = \langle T, \leq \rangle$ be a *time-frame* comprising a non-empty set T of times (moments, instants) on which $\leq$ is a reflexive and transitive ordering. The reflexivity of $\leq$ gives $\Diamond$ the Diodorean 'is or will be' interpretation. The frame $\mathbf{T}$ is directed, i.e. for all $t,s \in T$ there exists a $v \in T$ with $t \leq v$ and $s \leq v$ (any two elements have an upper bound).

Pursuing Smirnov's approach events would be identified with subsets of T. Let φ be a function assigning to event variable *a* a subset

$\varphi(a) \subseteq$ T (the set of times at which *a* takes place - an event history). The function φ will be extended in usual way:

$\varphi(a \cap b) = \varphi(a) \bigcap \varphi(b)$
$\varphi(a \cup b) = \varphi(a) \bigcup \varphi(b)$
$\varphi(\sim a) = \varphi(a)'$
$\varphi(\Diamond a) = \{x: \exists y(y \in \varphi(a) \text{ and } x \leq y)\}$ (*future history of event a*)

Here $\bigcap$, $\bigcup$, $'$ are the set-theoretic meet, join and complementation respectively.

The notion of truth would be described in a standard way:

$x \models_\varphi \theta a \Leftrightarrow x \in \varphi(a)$ (*an event occurred, is true in time x if and only if this time belongs to event history*)
$x \models_\varphi \alpha \vee \beta \Leftrightarrow x \models_\varphi \alpha$ or $x \models_\varphi \beta$
$x \models_\varphi \alpha \wedge \beta \Leftrightarrow x \models_\varphi \alpha$ and $x \models_\varphi \beta$
$x \models_\varphi \alpha \supset \beta \Leftrightarrow$ not $x \models_\varphi \alpha$ or $x \models_\varphi \beta$
$x \models_\varphi \neg\alpha \Leftrightarrow$ not $x \models_\varphi \alpha$
$x \models_\varphi \theta(\Diamond a) \Leftrightarrow x \in \varphi(\Diamond a) = \{z: \exists y(y \in \varphi(a) \text{ and } z \leq y)\}$ (*it is true in time x that event will be occurred if and only if this time belongs to the future history of the event*)

It is easily can be seen that with a help of standard methods we obtain an adequacy of combine logic with axioms A1-A2, B1-B4 with the semantics proposed.

Most interesting for us is the following peculiarity of such semantics. R.Goldblatt in [4] shows that an every frame $\mathbf{T}^n$ determines an S4.2 modal logic where $\mathbf{T}^n$ would be described in a following way.

Let $x=(x_1,\ldots,x_n)$ is an *n*-tuple of real numbers and $\mu(x)=x_1^2+x_2^2+\ldots + x_{n-1}^2 - x_n^2$. Then the frame $\mathbf{T}^n = \langle \boldsymbol{R}^n,\leq\rangle$ (where $\boldsymbol{R}^n$ is the set of all real *n*-tuples) will be *n*-dimensional apcetime ($n \geq 2$). For *x* and *y* in $\boldsymbol{R}^n$ we have

$$x \leq y \text{ iff } \quad \mu(y - x) \leq 0 \quad \text{and } x_n \leq y_n$$
$$\text{iff } \quad \sum_{i=1}^{n-1} (y_i - x_i)^2 \leq (y_n - x_n)^2 \quad \text{and } x_n \leq y_n$$

Such $\mathbf{T}^n$ is, in fact, a directed partially-ordered frame and Minkowski spacetime is $\mathbf{T}^4$. In last case the intended interpretation of $x \leq y$ is that a signal can be sent from 'event' *x* to 'event' *y* at a speed at most that of the speed of light, and so *y* is in the 'causal' future of *x* (assuming a choice of coordinates that gives the speed of light as one unit of distance per unit of time).The proof is based on the existence of *p*-morphism (a function, preserving ordering and directedness) from $\mathbf{T}^{n+1}$ into $\mathbf{T}^n$ (i.e. the "first" coordinate to be deleted).

Thus, an S4.2 modal algebra of events we need for our combine logic would be considered as a modal algebra $\mathbf{T}^{4+} = \langle \mathbf{M}, \cup, \cap, —, \Diamond \rangle$ where

(i) $\mathbf{M} = \boldsymbol{P}(\boldsymbol{R}^4)$ (a set of all subsets of $\boldsymbol{R}^4$);

(ii) $\cup, \cap, —$ are set-theoretical join, meet and complementation in $\mathbf{M}$;

(iii) for $A \in \mathbf{M}$ $\Diamond A = \{x: \exists y(y \in A \text{ and } x \leq y)\}$

if we use the standard Lemmon's method of obtaining modal algebra for a frame (cf. [7]).

And the other way round, our time-frame $\mathbf{T} = \langle T, \leq \rangle$ would be from the outset identified with $\mathbf{T}^4 = \langle \boldsymbol{R}^4, \leq \rangle$ and the function φ in a natural way will assign to an event $\Diamond a$ its 'causal' path or 'causal' future history in Minkowski spacetime.

In order to establish events-formulas connections in a more transparent way (e.g. in case of descriptions of something happened) we can enrich our language with an operator [-] ([α] means "the event, that α"). This involves an addition of the following axioms:

C1. $\theta[\alpha] \equiv \alpha$

C2. $\theta[\alpha \vee \beta] \equiv \theta([\alpha] \cup [\beta])$

C3. $\theta[\neg\alpha] \equiv \theta(\sim[\alpha])$

Clearly, the function φ ought also to be extended for terms of [-]-type:

$\varphi([\alpha]) = \{\mathrm{w}: w \models_{\varphi} \alpha\}$

Theorem 1. *Axioms PC+(A1-A2,B1-B4,C1-C3) are valid in the semantic above with the frame* $\mathbf{T}^4 = \langle \boldsymbol{R}^4, \leq \rangle$.

Proof is straightforward ■.

Approaching propositions and events as two different kind of entities we arrive at another semantics which is strictly algebraic exploiting a construction of algebraic bundle. In our case an algebraic bundle should be defined as a 4-tuple $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f}, \boldsymbol{g} \rangle$, where $\boldsymbol{A} = \langle A, +, — \rangle$ (the base) is a Boolean algebra (A contains two elements at least), $\boldsymbol{B} = \langle B, \oplus, ', \bullet \rangle$ is an S4.2-algebra, $\boldsymbol{f} : \boldsymbol{B} \to \boldsymbol{A}$, $\boldsymbol{g}: \boldsymbol{A} \to \boldsymbol{B}$ are embedding functions. Let 0, 1, $\circ$ and $\leq$ be in both algebras defined as usual. For $\boldsymbol{f}$ and $\boldsymbol{g}$ the following conditions are fulfilled:

$\boldsymbol{f}(k \oplus l) = \boldsymbol{f}(k) + \boldsymbol{f}(l)$,

$\boldsymbol{f}(k') = -\boldsymbol{f}(k)$,

$\boldsymbol{f}(\boldsymbol{g}(x)) = x$,

$\boldsymbol{g}(x + y) = \boldsymbol{g}(x) \oplus \boldsymbol{g}(y)$,

$\boldsymbol{g}(— x) = \boldsymbol{g}(x)'$,

where $x,y \in A$ and $k,l \in B$.

If F is a set of well-formed formulas and E is a set of events then a valuation $\boldsymbol{v}: F \bigcup E \to A \bigcup B$ is defined inductively as follows:

$\boldsymbol{v}(\alpha\vee\beta) = \boldsymbol{v}(\alpha) + \boldsymbol{v}(\beta)$,

$\boldsymbol{v}(\neg\alpha) = \text{—}\, \boldsymbol{v}(\alpha)$

(where α, β ***are*** wff and $\boldsymbol{v}(\alpha)$, $\boldsymbol{v}(\beta) \in A$),

$\boldsymbol{v}(a\cup b) = \boldsymbol{v}(a) \oplus \boldsymbol{v}(b)$,

$\boldsymbol{v}(\sim a) = \boldsymbol{v}(a)'$,

$\boldsymbol{v}(\Diamond a) = \bullet \boldsymbol{v}(a)$,

$\boldsymbol{v}(\theta a) = \boldsymbol{f}(\boldsymbol{v}(a))$,

$\boldsymbol{v}([\alpha]) = \boldsymbol{g}(\boldsymbol{v}(\alpha))$

(where a, b are events and $\boldsymbol{v}(a)$, $\boldsymbol{v}(b) \in B$).

Theorem 2. *Axioms PC+(A1-A2, B1-B4,C1-C3) are valid in any algebraic bundle* $\langle \boldsymbol{A},\boldsymbol{B},\boldsymbol{f},\boldsymbol{g}\rangle$.

Proof is straightforward ■.

This result is trivial enough but the following corollary shows that our algebraic bundle really would be a spacetime bundle with the fibres in $\mathbf{T}^{4+}$:

Corollary 1. *Axioms PC+(A1-A2,B1-B4,C1-C3) are valid in any algebraic bundle* $\langle \boldsymbol{A},\mathbf{T}^{4+},\boldsymbol{f},\boldsymbol{g}\rangle$.

Moreover, our bundle semantics should not be an algebraic one. We define a Kripke bundle as a 4-tuple $\langle \boldsymbol{W},\boldsymbol{E},\boldsymbol{f},\boldsymbol{g}\rangle$, where $\boldsymbol{W}$ and $\boldsymbol{E}$ are ordered sets $\boldsymbol{W} = \langle W, \leq\rangle$, $\boldsymbol{E} = \langle E, \leq\rangle$ while $\boldsymbol{f} : \boldsymbol{E} \to \boldsymbol{W}$, $\boldsymbol{g}: \boldsymbol{W} \to \boldsymbol{E}$ are surjective mappings. For $\boldsymbol{f}$ and $\boldsymbol{g}$ the following conditions are fulfilled:

1. for every $x,y\in\boldsymbol{E}$, $x\leq y$ implies $\boldsymbol{f}(x) \leq \boldsymbol{f}(y)$;
2. for every $x\in\boldsymbol{E}$ and every $w\in\boldsymbol{W}$, $\boldsymbol{f}(x) \leq w$ implies that there exists some $y\in\boldsymbol{E}$ with $x\leq y$ and $\boldsymbol{f}(y) = w$.

Again, let φ be a function assigning to event variable a a subset $\varphi(a)\subseteq \boldsymbol{E}$ and we extend function φ in usual way:

$\varphi(a\cap b) = \varphi(a) \bigcap \varphi(b)$

$\varphi(a\cup b) = \varphi(a) \bigcup \varphi(b)$

$\varphi(\sim a) = \varphi(a)'$

$\varphi(\Diamond a) = \{x: \exists y(y\in\varphi(a) \text{ and } x \leq y)\}$

where $\bigcap$, $\bigcup$, $'$ are the set-theoretic meet, join and complementation respectively.

A relation $\models$ is said to be a valuation on a Kripke bundle $\langle \boldsymbol{W},\boldsymbol{E},\boldsymbol{f},\boldsymbol{g}\rangle$ if it is a binary relation between each element $w\in\boldsymbol{W}$ and each atomic formula. We extend $\models$ inductively as follows:

$w \models \theta a \Leftrightarrow \varphi(a)\cap \boldsymbol{f}^{-1}(w) \neq \varnothing$

$w \models \alpha \vee \beta \Leftrightarrow w \models \alpha$ or $w \models \beta$

$w \models \alpha \wedge \beta \Leftrightarrow w \models \alpha$ and $w \models \beta$

$w \models \alpha \supset \beta \Leftrightarrow$ if $w \models \alpha$ then $w \models \beta$

$w \models \neg\alpha \Leftrightarrow$ not $w \models \alpha$

$w \models \theta(\Diamond a) \Leftrightarrow \varphi(\Diamond a) \cap \boldsymbol{f}^{-1}(w) \neq \varnothing$

A formula α is said to be valid in a Kripke bundle $\langle \boldsymbol{W},\boldsymbol{E},\boldsymbol{f},\boldsymbol{g}\rangle$ if for every valuation $\models$ on $\langle \boldsymbol{W},\boldsymbol{E},\boldsymbol{f},\boldsymbol{g}\rangle$ and every $w \in \boldsymbol{W}$, one has $w \models \alpha$.

From this definitions the next theorem easily follows.

Theorem 3. *Axioms PC+(A1-A2, B1-B4,C1-C3) are valid in any Kripke bundle* $\langle \boldsymbol{W},\boldsymbol{E},\boldsymbol{f},\boldsymbol{g}\rangle$.

Corollary 2. *Axioms PC+(A1-A2,B1-B4,C1-C3) are valid in a Kripke bundle* $\langle \boldsymbol{W},\mathbf{T}^4,\boldsymbol{f},\boldsymbol{g}\rangle$.

3. Orthomodular Combined Causal Logic of Minkowski Spacetime

Now we shall consider system of combined logic with the following axioms which one should add to the axiom schemata of classical sentential logic and the rule *modus ponens*:

C1. $\theta a \vee \theta b \equiv \theta(a \cup b)$
C2. $\theta a \wedge \theta b \equiv \theta(a \cap b)$
C3. $\theta(\sim a \cap a) \supset \theta b$
C4. $\theta b \supset \theta(\sim a \cup a)$
C5. $\theta(\sim\sim a) \equiv \theta a$
C6. $\theta(\sim(a \cap b) \equiv \theta(\sim a \cup \sim b)$
C7. $\theta(\sim(a \cup b) \equiv \theta(\sim a \cap \sim b)$
C8. $(\theta a \supset \theta b) \supset (\theta(a \cup (\sim a \cap b)) \equiv \theta b)$

One can easy come to the conclusion that C1-C7 stand for the axioms of an ortholattice while C1-C8 stand for the axioms of an orthomodular lattice (cf. [1]). Thus, a system with axioms *PC+*(*C1-C8*) is a system of combined logic with an orthomodular lattice as its internal logic.

Kripkean semantics for such a logic we obtain using the notion of orthoframe and orthocomplement in and orthoframe [3, p. 432]:

Definition. An orthoframe is a relational structure $\mathbf{W} = \langle W, R\rangle$ where W is non-empty set of worlds and the accessibility relation R is a binary reflexive and symmetric relation on W. For any set of worlds $X \subseteq W$ the orthocomplement $X^{\perp}$, of X is defined as follows:

$X^{\perp} = \{w : \forall v(v \in X \Rightarrow \text{not } Rvw\}$

In other words, $X^{\perp}$ is the set of worlds which are unaccessible to all elements of X.

Let φ be a function assigning to event variable a a subset $\varphi(a) \subseteq W$ (the set of worlds at which a takes place - an event history) and the following condition is filfilled:

(i) $\forall w(w \in \varphi(a) \text{ iff } \forall v(Rwv \Rightarrow v \notin \varphi(a)^{\perp})$.

Corollaries of such a definition are as follows [3, p.433]:

(ii) $\forall w(w \notin \varphi(a) \Rightarrow \exists v(Rwv \text{ and } v \in \varphi(a)^{\perp})$,

(iii) $\varphi(a) = \varphi(a)^{\perp\perp}$.

In order to transform orthoframe into orthomodular one we need the function φ satisfies the following condition (orthomodularity property) [3, p.437]:

(iv) $\varphi(a) \not\subset \varphi(b) \Rightarrow \varphi(a) \cap (\varphi(a) \cap \varphi(b))^{\perp} \neq \varnothing$

The function φ will be extended in usual way:

$\varphi(a \cap b) = \varphi(a) \cap \varphi(b)$
$\varphi(a \cup b) = \varphi(a) \cup \varphi(b)$
$\varphi(\sim a) = \varphi(a)^{\perp}$

The notion of truth would be now introduced in a standard way:

$x \models_{\varphi} \theta a \Leftrightarrow x \in \varphi(a)$
$x \models_{\varphi} \alpha \vee \beta \Leftrightarrow x \models_{\varphi} \alpha$ or $x \models_{\varphi} \beta$
$x \models_{\varphi} \alpha \wedge \beta \Leftrightarrow x \models_{\varphi} \alpha$ and $x \models_{\varphi} \beta$
$x \models_{\varphi} \alpha \supset \beta \Leftrightarrow$ not $x \models_{\varphi} \alpha$ or $x \models_{\varphi} \beta$
$x \models_{\varphi} \neg\alpha \Leftrightarrow$ not $x \models_{\varphi} \alpha$

If we enrich our language with the operator [-] then we need to add the following axiom schemata to (C1-C8)-list of axioms:

D1. $\theta[\alpha] \equiv \alpha$
D2. $\theta[\alpha \vee \beta] \equiv \theta([\alpha] \cup [\beta])$
D3. $\theta[\alpha \wedge \beta] \equiv \theta([\alpha] \cap [\beta])$
D4. $\theta[\neg\alpha] \equiv \theta(\sim[\alpha])$

Clearly, the function φ ought also to be extended for terms of [-]-type:

$\varphi([\alpha]) = \{w: w \models_{\varphi} \alpha\}$

From this definitions the next theorem easily follows.

Theorem 3. *Axioms PC+(C1-C8,D1-D4) are valid in any Kripke realization* $\langle \mathbf{W}, \models_{\varphi} \rangle$.

The most interesting for us will be the case if we use W.Cegla's orthomodular lattice $L(X,\perp)$ (for which the condition $l(X,\perp) = L(X,\perp)$ is satisfied) in a role of our set of values of the function φ, i.e. when $\mathbf{W} = L(X,\perp)$. Now X can be identified with Minkowski spacetime $\mathsf{M} = \boldsymbol{R} \times \boldsymbol{R}^3$ with scalar product $x*y = - x_0y_0 + \bar{x}\,\bar{y}$ [2, p. 422]. Two points $x,y \in \mathsf{M}$ are orthogonal iff

$|x_0 - y_0| \leq (1/\alpha) \| \bar{x} - \bar{y} \|$

and every maximal orthogonal set is given by the function g: $\boldsymbol{R}^3 \rightarrow \boldsymbol{R}$

such that

$$|\, g(\bar{x}) - g(\bar{y})| \leq (1/\alpha) \,\|\, \bar{x} - \bar{y} \,\|$$

while if $\alpha = 1$ then the orthogonality relation means that x is a space or like-light to y.

An algebraic bundle now should be defined as a 4-tuple $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f}, \boldsymbol{g} \rangle$, where $\boldsymbol{A} = \langle A, +, \circ, - \rangle$ (the base) is a Boolean algebra (A contains two elements at least), $\boldsymbol{B} = \langle B, \oplus, \otimes, {}^{\perp} \rangle$ is an orthomodular lattice, $\boldsymbol{f} : \boldsymbol{B} \to \boldsymbol{A}$, $\boldsymbol{g}: \boldsymbol{A} \to \boldsymbol{B}$ are embedding functions. Let 0, 1, $\circ$ and $\leq$ be in both algebras defined as usual. For $\boldsymbol{f}$ and $\boldsymbol{g}$ the following conditions are fulfilled:

$\boldsymbol{f}(k \oplus l) = \boldsymbol{f}(k) + \boldsymbol{f}(l)$,
$\boldsymbol{f}(k \otimes l) = \boldsymbol{f}(k) \circ \boldsymbol{f}(l)$,
$\boldsymbol{f}(k^{\perp}) = -\boldsymbol{f}(k)$,
$\boldsymbol{f}(\boldsymbol{g}(x)) = x$,
$\boldsymbol{g}(x + y) = \boldsymbol{g}(x) \oplus \boldsymbol{g}(y)$,
$\boldsymbol{g}(-x) = \boldsymbol{g}(x)^{\perp}$,

where $x, y \in A$ and $k, l \in B$.

If F is a set of well-formed formulas and E is a set of events then a valuation $\boldsymbol{v}$: $F \bigcup E \to A \bigcup B$ is defined inductively as follows:

$\boldsymbol{v}(\alpha \vee \beta) = \boldsymbol{v}(\alpha) + \boldsymbol{v}(\beta)$,
$\boldsymbol{v}(\alpha \wedge \beta) = \boldsymbol{v}(\alpha) \circ \boldsymbol{v}(\beta)$,
$\boldsymbol{v}(\neg \alpha) = -\boldsymbol{v}(\alpha)$

(where α, β are wff and $\boldsymbol{v}(\alpha)$, $\boldsymbol{v}(\beta) \in A$),

$\boldsymbol{v}(a \cup b) = \boldsymbol{v}(a) \oplus \boldsymbol{v}(b)$,
$\boldsymbol{v}(a \cap b) = \boldsymbol{v}(a) \otimes \boldsymbol{v}(b)$,
$\boldsymbol{v}(\sim a) = \boldsymbol{v}(a)^{\perp}$,
$\boldsymbol{v}(\theta a) = \boldsymbol{f}(\boldsymbol{v}(a))$,
$\boldsymbol{v}([\alpha]) = \boldsymbol{g}(\boldsymbol{v}(\alpha))$

(where a, b are events and $\boldsymbol{v}(a)$, $\boldsymbol{v}(b) \in B$).

Theorem 2. *Axioms PC+(C1-C8,D1-D4) are valid in any algebraic bundle* $\langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f}, \boldsymbol{g} \rangle$.

Proof is straightforward ■.

Again, this result is trivial enough but the following corollary shows that our algebraic bundle really would be (Minkowski spacetime) bundle with the fibres in $L(\mathsf{M}, \perp)$.

Corollary 3. *Axioms PC+(C1-C8,D1-D4) are valid in any algebraic bundle* $\langle \boldsymbol{A}, L(\mathsf{M}, \perp), \boldsymbol{f}, \boldsymbol{g} \rangle$.

In an obvious way we can extend this result on a respective Kripke bundle.

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