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GAMES WITH COMMON BELIEF ON PAYOFF FUNCTION

I. Introductory Remarks and Definitions

To introduce a definition of a game in normal form, we need the following concepts: the set I of players; the (pure) strategy sets S_I for each agent $i \in I$; the payoff functions p_i $(s_1, \dots, s_I)$ that assign payoffs for all $i \in I$ as determined by a possible strategy combination $(s_1, \dots, s_I)$.

Definition 1. A *game* Γ *in normal form* is a triple $(I,(S_i)_{i \in I},(p_i)_{i \in I})$.

A proposition P is *common knowledge* if all players know that P, and know that all players know that P, and know that all players know that all players know that P, and so on up to any degree of iteration. A proposition P is *common belief* if all players believe that P, and it is common knowledge that all players believe that P.

Consider a finite set of propositions $P = \{P_1, \dots, P_n\}$. The probability distribution μ on P is called *probabilistic common belief* if for all players, their doxastic attitude to $P_1, \dots, P_n$ is governed by μ, and this is common knowledge. Consider now a game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$, and a finite set $B = \{(p^1{}_i)_{i \in I}, \dots, (p^n{}_i)_{i \in I}\}$ such that for each $1 \le j \le n$ and for each $i \in I$, $(p^j{}_i)$ is an (alternative) payoff function $(p^j{}_i)$ $(s_1, \dots, s_I)$.

Definition 2. Given $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ and $B = \{(p^1{}_i)_{i \in I}, \dots, (p^n{}_i)_{i \in I}\}$, $\Delta = (\Gamma, \mu[B])$ is a *game with common belief on payoff function (g.c.p.b.),* where $\mu[B]$ is a probability distribution on B.

Given a g.c.p.b. $\Delta = (\Gamma, \mu[B])$, we will call Γ *the ontological component* of Δ, and $\mu[B]$, its *doxastic component*, their intended interpretation being that (i) the players are *really* in the situation determined by $(p_i)_{i \in I}$; and (ii) their probabilistic *common belief* about what is their situation like is (represented by) $\mu[B]$.

Note that standard games in normal form constitute (or, better, correspond to) a proper subclass of all g.c.p.b.'s. To be (or: to correspond to) a standard game in normal form, a g.c.p.b., $\Delta = (\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I}), \mu[B])$, has to meet the two following conditions:

(i) $(p_i)_{i \in I} \in B$;
(ii) $\mu((p_i)_{i \in I}) = 1$;
for any $1 \le j \le n$,
$\mu((p^j{}_i)_{i \in I}) = 0$ if $(p^j{}_i)_{i \in I} \ne (p_i)_{i \in I}$.

In theory of games with common payoff beliefs, like anywhere else in game theory, the central notion is that of a solution of the game at issue. But with g.c.p.b.'s, unlike anywhere else in game theory, at work is a distinction between the players' subjective image of the game and the game as it is in reality. The players deliberate and act upon their common beliefs about their situation; and when they have reached the outcome of their combined action, they may be surprised at the value of their payoffs, if the image they were acting upon was distorted. The solution of a g.c.p.b. is a function of the players' subjective beliefs, but the resulting payoffs are a function of both the beliefs and the actual reality.

Note that, generally speaking, the concepts involved in a definition of game solution may differ across classes of games: from maximin strategies for a two-player zero-sum game, via dominant strategy equilibrium, via iterated dominance equilibrium, to Nash equilibrium, and the many kinds of its refinement. The bulk of theory of g.c.p.b.'s may be applied to the concept of game solution as construed in terms of any of the above-listed notions. Nevertheless, for the sake of definiteness, we will presume in what follows that each time we consider a game with a unique Nash equilibrium, the concept of game solution should be understood as resulting from the notion of Nash equilibrium.

We are now in a position to construct a definition of *doxastic solution of a g.c.p.b. under doxastic certainty*:

Definition 3. Given a g.c.p.b. $\Delta = (\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I}), \mu[B])$, such that for all $(p^j{}_i)_{i \in I} \in B$, except one, namely: $(p^J{}_i)_{i \in I}$, $\mu(p^j{}_i) = 0$, *doxastic solution* for Δ is the solution of game

$\Gamma^* = (I, (S_i)_{i \in I}, (p^J{}_i)_{i \in I})$.

To define the concept of doxastic solution in a general case, consider a game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ whose payoff function maps strategy combinations to *expected payoff values*, relative to a probability distribution, rather than to just the numerical values of the payoffs. Such a game will induce a notion of a game solution exactly in the same way in which a standard (matrix) game does, - because after the expected payoff values have been calculated, the payoff function will have mapped strategy combination to numerical values, exactly in the same way in which the payoff function in a standard game does.

Definition 4: Doxastic solution of a g.c.p.b. (general case). Given a g.c.p.b. $\Delta = (\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I}), \mu[B])$, denote the numerical payoff function resulting from probability distribution $\mu[B]$, $(p(\mu[B])_i)_{i \in I}$. *Doxastic solution* of Δ is the solution of game $\Gamma^* = (I, (S_i)_{i \in I}, (p(\mu[B])_i)_{i \in I})$.

II. Comparing doxastic and ontological payoffs

Definition 5. If a g.c.p.b. $\Delta = (\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I}), \mu[B])$ has a doxastic solution $(s_1, \ldots, s_I)$, *doxastic payoff combination* for Δ is the combination $(p_1(s_1, \ldots, s_I), \ldots, p_I(s_1, \ldots, s_I))$.

Definition 6. Given a g.c.p.b. $\Delta = (\Gamma, \mu[B])$, if Γ has a solution $(s_1, \ldots, s_I)$, we will call it *ontological solution* of Δ, and the combination $(p_1(s_1, \ldots, s_I), \ldots, p_I(s_1, \ldots, s_I))$, *ontological payoff combination* for Δ.

Definition 7. A g.c.p.b. $\Delta = (\Gamma, \mu[B])$ is *doxastically negative (neutral, positive, incomparable)*, if its doxastic payoff combination is Pareto-dominated by (respectively: coincides with, Pareto-dominates, is Pareto-incomparable with) its ontological payoff combination.

Definition 8. Given a game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ in normal form and a set B of possible alternatives to $(p_i)_{i \in I}$, *the* $<\Gamma, B>$*-class* is the class of all g.c.p.b.'s of the form $\Delta = (\Gamma, \mu[B])$ with $\mu[B] \in M[B]$, where M[B] is the set of all probability distributions over B.

A $<\Gamma, B>$-class can be usefully partitioned into four domains: (i) *negative*; (ii) *neutral*; (iii) *positive*; and (iv) *incomparability domain*, according to whether the members of a domain are doxastically negative, neutral, positive or incomparable, respectively. An intuitive significance of such a partition should be clear: we are interested in the question of what happens if the players move from their current (perhaps, distorted and/or incomplete) image of the game to its correct and complete image.

To the four domains, there correspond the four following answers in the same order:

– each player would have their solution payoff strictly increased;
– all the solution payoffs would remain the same;
– each player would have their payoff strictly diminished;
– the result would differ for different players.

III. The issue of the emptiness of the positive domain

Of course the most surprising case would be there, if we find the positive domain non-empty. On the first glance such a case seems counter-intuitive: How can it be that correct and complete knowledge of the situation can do harm to **all** the players? Is not knowledge always power – at least for some of them?

To give one interesting answer to the above questions, we need a definition of Pareto-suboptimality:

Definition 9. A game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ in normal form is *Pareto-suboptimal*, if (i) Γ has a unique Nash equilibrum $(s_1, \ldots, s_I)$, and (ii) there is a combination of strategies $(s^*_1, \ldots, s^*_I)$ such that for every $i \in I$, $p_i (s^*_1, \ldots, s^*_I) > p_i (s_1, \ldots, s_I)$.

Now it is straightforward to demonstrate the following fact:

Fact: For any Pareto-suboptimal game $\Gamma = (I, (S_i)_{i \in I}, (p_i)_{i \in I})$ in normal form, there exists a set B of possible alternatives to $(p_i)_{i \in I}$ such that the positive domain of $<\Gamma, B>$-class is not empty.

Proof: Let $B = \{(p^*_i)_{i \in I}\}$, where $(p^*_i)_{i \in I}$ is determined by the following conditions:

(I) for every $i \in I$, $p^*_i (s^*_1, \ldots, s^*_I) = 1$;

(II) for every $i \in I$, $p^*_i (\pi) = 0$, where (π) is a combination of strategies other than $(s^*_1, \ldots, s^*_I)$.

It is immediately obvious that, given such Γ *and B, the* $<\Gamma, B>$*-class has exactly one member, and that member belongs to its positive domain.*

Given the above fact, the framework of g.c.p.b.'s may be helpful for the task of investigating positive effects of shared epistemic imperfections in the context of Pareto-suboptimal interactive situations – e.g., the Prisoners' Dilemma, the problem of collective action, and the like.