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WHAT IS PROPOSITIONAL CLASSICAL LOGIC? (A Study in Universal Logic)[2]

Abstract. *The aim of this paper is to try to characterize classical propositional logic (CPL) with the notion of mathematical structure.*

We start by justifying this approach. We recall the importance and significance of the notion of structure in mathematics and in logic. We explain the idea of a general theory of logics based on structures, Universal Logic.

CPL is not one structure but a class of equivalent structures, CPL-structures. We survey a series of structures that can be considered as CPL-structures. The main problem is to find a notion of equivalence which permits to gather into a whole this multiplicity.

We show in particular that the modern concept of equivalence of structures, based on the notion of expansion by definition and isomorphism, is not adequate to define a satisfactory notion of equivalence that will define the class of CPL-structures. An alternative definition, postmodern equivalence, is introduced.

It appears that this tentative of characterization of the class of CPL-structures is not only relevant for Universal Logic, but also for the general theory of mathematical structures, since the case of CPL-structures shows the insufficiency of the modern concept of equivalence between structures.

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1. Classical propositional logic, Universal Logic and the theory of structures

Logic is nowadays a very vast and varied field and probably there is no logician who has a global knowledge of it, from inaccessible cardinals to lambda calculus, from cylindric algebra to the hyperarithmetical hierarchy, from stability theory to quantum logic. On the other hand, there is a part of logic that every logician certainly knows, it is classical propositional logic (CPL hereafter).

So it seems that the question "What is classical propositional logic?" does not have to be asked since everybody knows of course what is CPL. But there are several ways of "knowing". In some sense everyone knows what is a cat, a number or light, mainly because one is acquainted with these things. But few people, if any, are able to give proper definitions of a cat, a number or light. To give such definitions people turn either to philosophy or science.

Maybe people think that CPL is already scientific and well defined. So they think that someone who asks "What is CPL?" can only be a helpless philosopher. But we must remember that some hundred years ago mathematicians thought that people like Frege and Russell were helpless philosophers.

To the question "What is CPL?", a bright young logician could answer: "Very easy, two-valued truth-tables are CPL, such or such Hilbert-type axiomatic system is CPL, such or such Gentzen-type proof system is CPL, such or such tableau system is CPL, Boolean algebra is CPL, all this and many more is CPL." We can reply to this bright young logician in the same way as Socrates to Theaetetus: "How generous and open-handed of you! You were asked for one thing, but you're offering several, and a variety instead of something simple" (Plato, Theaetetus, 146d).

One way to give a definition by comprehension of CPL rather than by extension, is to consider CPL as a structure. This is useful in practice: one can present the completeness theorem saying that two-valued truth-tables and a given Hilbert-type axiomatic system define the same structure. As it is known, the notion of structure has been crucial to reorganized and unified mathematics and is the cornerstone of modern mathematics[3]. Following the structuralist approach to mathematics, a general theory of logics has been developed under the name Universal Logic. It is a theory of logical structures in the same

[3] There are a lot of interesting philosophically oriented papers by Bourbaki and his friends about the notion of mathematical structure. Most of these papers are difficult to find and not very well-known. The reader may have a look at the most famous [3]. He may also consult the historical book of Corry [6].

way as Universal Algebra is a theory of algebraic structures (see [1], [2])[4].

But what is a logical structure? That is the question. In fact we will see in this paper that it is even difficult to answer this question for the most well-known logical structure, i.e. CPL.

The problem is that CPL does not appear as one structure but as a multiplicity of structures, CPL-structures. Then if we want to properly define CPL according to the structural approach of Universal Logic, we must find a way to unify this multiplicity into a whole, through a relation of equivalence between structures.

As it is known, a Boolean algebra can be represented as a Boolean ring or as a distributive complemented lattice. These two structures are equivalent. Boolean algebra is a class of equivalent structures, and a Boolean algebra is any member of this class. Everything is clear and there is no problem at all.

But the difficulty with CPL-structures is that the standard notion of equivalence between structures does not applied straightforwardly to it. Equivalence between structures is a fundamental notion of the general theory of structures and CPL-structures happen to challenge it. If we consider the general theory of structures as part of the foundation of mathematics (conceptual foundation by opposition to the axiomatic approach, cf [8]), we can say that Universal Logic, through the simple case of CPL, challenges the foundation of mathematics.

After presenting the standard notion of equivalence and the concepts of modern mathematical structuralism on which it is based, we present a series of different CPL-structures. The difficulty to apply the standard notion of equivalence to these structures leads us to develop an alternative notion of equivalence based on postmodern structuralism.

2. Modern mathematical structuralism

Modern mathematical structuralism has to be understood by contrast with *post-modern* mathematical structuralism that we will introduce later on, also by contrast with modern *logical* structuralism. Logical structuralism has been developed within model theory. It is

[4] The structuralist approach in logic can be traced back to Tarski (cf [13]), it has been further developed in Poland especially by Suszko's group in the sixties and the seventies (cf. [4] [5]) and more recently by the Barcelona logic group [7]. People like Suszko and his followers tend to consider logical structures as special cases of algebraic structures. Whether this view is correct or not depends on some points which will be discussed in the present paper. Our idea is to consider Universal Logic as independent of Universal Algebra, rather than a part of it. This view was already held by Porte ([11]). His idea was that logical structures were fundamental mother structures in the sense of Bourbaki.

fundamentally dicephalous, there is on one side the *reality* of the structure, on the other side the *language* of the structure. In mathematical structuralism we will consider only the "reality" of the structure[5]. A structure is an object of the following type:

$$S = \langle \mathbf{S}; \tau \rangle$$

S is called the *domain* of the structure, it is just a set.
τ is called the *type* of the structure, it is a sequence of relations defined on the domain (we take the concept of relation here in a very broad sense: including functions and not only relations between elements of the domain, but relations between parts of the domain, etc.)

We say that two structures have the same type iff their types have the same quantity and quality, that is to say if the lengths of the sequences are the same and if the first members of the two sequences are similar constructions (e.g. two binary relations defined on the power set of the domain), the second members also, etc. Now we will introduce several notions that will enable us to define the notion of equivalence between structures.

Definition of isomorphism

The relation of *isomorphism* is defined between structures of the same type. Two structures are isomorphic iff there is a bijection between them that preserves all the relations of the types. One may think that the notion of isomorphism is the basic notion of identity or equivalence between structures. But this notion is too strong for two reasons: one may want to identify structures of *different cardinalities* (cf. section 8) and one may also want to identify structures of *different types*.

Definition of expansion/reduct

A structure $S2 = \langle \mathbf{S}2; \tau 2 \rangle$ is an expansion of a structure $S1 = \langle \mathbf{S}1; \tau 1 \rangle$ (or $S1$ is a reduct of $S2$) iff

- they have the same domain: $\mathbf{S}1 = \mathbf{S}2$
- the type of $S1$ is included in the type of $S2$: $\tau 1 \subseteq \tau 2$

A typical example is: $\langle \mathbf{N}; +, \times, \leq \rangle$ is an expansion of $\langle \mathbf{N}; +, \leq \rangle$

Definition of definition by expansion

If the new elements of the type of $S2$ can be defined in terms of the elements of the type of $S1$, we say that $S2$ is an *expansion by definition* of $S1$.

[5] The definitions that we present here are adaptations of definitions of logical structuralism, this adaptation is probably an original contribution that we are making here.

The difficulty here is to define definition. One can admit several kinds of definitions, and accordingly a structure is or is not expansion by definition of a given structure. Let us give a typical example. $<\mathbf{N}; 0, s, +, \times, \leq >$, where s is the *successor function*, is an extension by definition of $<\mathbf{N}; 0, s>$ if we consider second-order definitions (i.e. definitions involving quantification over parts of the domain of the structure), but it is not an extension by definition if we restrict ourselves to first-order definitions.

Definition of equivalence between structures

Two structures are equivalent iff they have a common (up to isomorphism) expansion by definition.

In particular if a structure is an expansion by definition of another one, they are both equivalent. The problem of which kind of definitions we will admit is therefore also fundamental here. For example, from a second-order viewpoint $<\mathbf{N}; 0, s, +, \times, \leq >$ is equivalent to $<\mathbf{N}; 0, s>$ but not from a first-order viewpoint.

Another important notion of mathematical structuralism is the notion of extension. It is not needed to define equivalence in the modern sense, but we will need it to define equivalence in the postmodern sense.

Definition of extension/substructure

A structure $\mathcal{S}2 = <\mathbf{S}2; \tau 2>$ is an *extension* of a structure $\mathcal{S}1 = <\mathbf{S}1; \tau 1>$ (or $\mathcal{S}1$ is a *substructure* of $\mathcal{S}2$) iff

- the domain of $\mathcal{S}1$ is included in the domain of $\mathcal{S}2$: $\mathbf{S}1 \subseteq \mathbf{S}2$
- they have the same type
- each element of $\tau 1$ is a restriction of the corresponding element of $\tau 2$ to $\mathbf{S}1$

A typical example is: $<\mathbf{Z}; 0, s, +, \times, \leq >$ is an extension of $<\mathbf{N}; +, \times, \leq >$.

Note that in this example we use the same symbol + to denote two different things (the same for $\times$ and $\leq$). As Bourbaki would say "C'est un abus de langage dont les mathematiciens sont tres friands". In fact such "friandise" is exactly justified and explained by the definition of extension of structure. Mathematicians use the same symbol because although these two symbols denote two different functions, they denote functions which are intimately connected.

3. Relation of consequence or operator of consequence?

The following structure will be the structure of reference for our investigations about CPL-structures:

$$K = <\mathbf{F}; \neg, \wedge, \vee, \rightarrow, |\!-\!>$$

Elements of **F** are called *formulas*. The set **F** is freely generated from a subset **A** of *atomic formulas* with the unary function ¬, called negation, and three binary functions: ^, called conjunction, ∨, called disjunction, and →, called implication. That is to say the following structure is an absolutely free algebra:

$$F = <\mathbf{F}; \neg, \wedge, \vee, \rightarrow>$$

The structure K can therefore be represented in a simpler way as follows:

$$K = <F; |\!-\!>$$

The relation |- is called *relation of consequence*. It is a relation between sets of formulas and formulas, i.e. $|\!- \subseteq P(\mathbf{F}) X \mathbf{F}$.

There are thousands of structures that one may want to consider as CPL-structures. And for each one we must check if the standard notion of equivalence apply in such a way that this given structure is equivalent to *K*.

Let us start with an easy case. Instead of considering a *relation* of consequence |- to define CPL, we can consider, as Tarski did, an *operator* of consequence *Cn*. It is easy to see that it is the same, that we have here two equivalent structures.

Consider the structure

$$Kn = <F; Cn>$$

where *Cn* is a function from **F** to **F**.

It is not difficult to show that *Kn* and *Cn* are equivalent. We consider the following structure:

$$KKn = <F; |\!-; Cn>$$

This structure is an expansion by definition of *K* and of *Cn*. The following definition is used:

$$a \in CnT \text{ iff } T \,|\!-\, a$$

4. Consequence or tautologies?

Let us now turn to a more complex case. Consider the structure

$$T = <F; \mathbf{T}>$$

where **T** is the subset of **F**, the set of classical *tautologies*.

We can say that the following structure

$$KT = <F; |\!-; \mathbf{T}>$$

is an expansion by definition of *K* and *T* using respectively the two following definitions:

(1) a ∈ **T** iff ∅ |- a

(2) T |- b iff there exists elements a_1, ... a_n of T, such that $a_1 \wedge \ldots \wedge a_n \rightarrow b \in \mathbf{T}$

In this sense *T* and *K* are equivalent structures.

However there is a fundamental property that distinguishes *T* and *K*. The first is decidable (in the sense that the set **T** is recursive) and the second is not decidable (in the sense that the relation |- is not recursive).

At this point it is important to note that what we want to establish is a congruence relation between structures. Our notion of equivalence will be a congruence relation depending on which *superstructure* we will consider, which properties will be taken into account in this superstructure. If we think that *T* and *K* must not be considered as equivalent, then we just take a superstructure featuring decidability. Or we may take the other option.

Another way to look at the problem is to analyze the definition (2) above. If one doesn't want *T* and *K* to be equivalent, one can find a way to rule out this definition. On the other case one has to be able to justify the allowance of this definition (and therefore all similar ones) and has to be ready to accept all the consequences of such kind definition.

We consider now the structure

$$M = <F; |-_{m}>$$

where $|-_{m}$ is a relation between sets of formulas, i.e. $|-_{m} \subseteq P(\mathbf{F})XP(\mathbf{F})$

We can say that the following structure

$$KM = <F; |-; |-_{m}>$$

is an expansion by definition of *K* and *M* using respectively the two following definitions:

(1) T $|-_{m}$ U iff there exists $a \in$ U such that T |- a

(2) T |- a iff T $|-_{m}$ $\{a\}$

Let us also consider the structure

$$M_0 = <F; |-_{m0}>$$

where $|-_{m0}$ is a relation between finite sets of formulas.

Structures of type *M* or M_0 have been considered in the literature, sometimes under the name multi-conclusion logic (cf. [12]). The problem is that people don't always made a sharp distinction between *M* and M_0. But the difference between these two structures is the same as between *K* and *T*, since one is decidable and not the other. So the equivalence between them is not straightforward.

5. Or or and?

Classical logic can be presented in a more economical way that *K* since implication and disjunction are definable with negation and conjunction. One could also consider classical logic as a structure with only one connective, like the Sheffer's stroke. If we want to be generous instead of being stingy, we could on the other hand consider a richer structure than K, adding for example the bi-implication $\leftrightarrow$, etc. Generally all these ways of presenting CPL are considered as equivalent. Let us call this intuitive notion of equivalence, *language-equivalence*, and let us see if we can define language-equivalence within classical structuralism.

Consider the structures

$$K_{and} = <F_{and}; |\!-_{and}>$$
$$K_{or} = <F_{or}; |\!-_{or}>$$

where F_{and} and F_{or} are respectively the structures

$$<\mathbf{F}_{and}; \neg_{and}, \wedge_{and}, \rightarrow_{and}>$$
$$<\mathbf{F}_{or}; \neg_{or}, \vee_{or}, \rightarrow_{or}>$$

where x_{and} are the restrictions of x to $\mathbf{F}_{and}$
and x_{or} are the restrictions of x to $\mathbf{F}_{or}$.

Everybody knows that it is possible to define $\vee$ with $\neg$ and $\wedge$ in the following way:

$$a \vee b \equiv_{Def} \neg(\neg a \wedge \neg b)$$

but K is not an expansion by definition of K_{and} simply because K is not an expansion of K_{and}: they have not the same domain.

K looks like an extension of K_{and} since $\mathbf{F}_{and} \subseteq \mathbf{F}$ and $*_{and}$ are restrictions of x to $\mathbf{F}_{and}$ but K is not strictly speaking an extension of K_{and} since these two structures have not the same type: $\vee$ is not part of the type of K_{and}.

It seems that classical structuralism, based on extension, expansion and isomorphism, is not enough to analyze the relations between K and K_{and}, and more generally the language-equivalence relation. Let us turn therefore into something more sophisticated.

6. Postmodern mathematical structuralism

Following postmodernism (cf. the distinction between differ*e*nce and differ*a*nce), we introduce the notions of expension, extansion and their correlates.

An *expension* is an expansion of an extension. An *extansion* is an extension of an expansion. We define then in a natural way the notions of *expension by definition and extansion by definition*. And we say that two structures are *postmodern equivalent* iff they have a common (up to isomorphism) extansion or expension by definition.

K is an extansion by definition of K_{and}, since

$$K = <\mathbf{F}; \neg, \wedge, \vee, \rightarrow, |\!->$$

is an extension of

$$<\mathbf{F}_{and}; \neg_{and}, \wedge_{and}, \vee_{and}, \rightarrow_{and}, |\!-_{and}>$$

which is itself an expansion by definition of

$$K = <\mathbf{F}_{and}; \neg_{and}, \wedge_{and}, \rightarrow_{and}, |\!-_{and}>$$

That seems fine: the two structures K and K_{and} are therefore equivalent in the postmodern sense. And the postmodern notion of equivalence also resolves the problem of all language-equivalent structures. But maybe it solves also too much problems, in the sense that it mixes bananas with tomatoes, it puts in the same equivalence class, structures that are too much different.

It is too early to claim that postmodernism structuralism will overthrow modernism structuralism, the notion of postmodern equivalence has to be investigated more systematically.

7. Logic or algebra?

Maybe there is a way to solve the problem of language-equivalence without going into postmodern structuralism, this is the algebraic way.

Given the structure K we consider the binary relation -||- on F defined as follows:

$$a \;-\!||\!-\; b \text{ iff } \{a\} \;|\!-\; b \text{ and } \{b\} \;|\!-\; a$$

It is known that this relation is a congruence relation. Factorizing K with this relation we get the following structure:

$$K^a = <\mathbf{F}^a; \neg^a, \wedge^a, \vee^a, \rightarrow^a, |\!-^a>$$

We can also factorize the structure K_{and} by the corresponding congruence relation $-\!||\!-_{and}$ and then get the following structure:

$$K^a_{and} = <\mathbf{F}^a_{and}; \neg^a_{and}, \wedge^a_{and}, \rightarrow^a_{and}, |\!-^a_{and}>$$

It is easy to see that $\mathbf{F}^a = \mathbf{F}^a_{and}$ so K^a and K^a_{and} have the same domain, and K^a is an expansion by definition of K^a_{and}.

If one consider that K is equivalent to K^a and that K_{and} is equivalent to K^a_{and} then, due to the above fact, K is equivalent to K_{and} and we have solve the problem of language-equivalence.

But can we consider that a structure is equivalent to one of its factorization? Here we have a problem similar to the one of section 4. A structure and one of its factorization don't have necessary the same properties. So if we want our equivalence here to be a congruence relation, we have to consider that the properties that distinguish a structure and its factorization are not essential.

This problem is related to the question of the distinction between algebra and logic, between Universal Algebra and Universal Logic. Can we consider that CPL is not less, not more than Boolean algebra, and that all the work that has been carried out between Boole and Tarski[6] was meaningless?

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[6] In [14], Tarski proved that we get a Boolean algebra by factorizing CPL.

At this point it is important to emphasize that the difference between CPL, as represented by *K*, and a Boolean algebra is not only a question of factorization. The structure K^a is not strictly speaking a Boolean algebra. The difference between K^a and a Boolean algebra is similar to the difference between *K* and *T*. In a Boolean algebra generally we consider a binary relation (identity) between objects of the domain and not between sets of objects and objects, like $|\!-^{a}$. If we consider CPL as *T*, then the difference between CPL and a Boolean algebra is just a question of factorization (in this case the relation of congruence has to be defined with the implication – cf. Tarski's original definition).

Nowadays most people would rather consider a logic as a structure of the type of *K*, than of the type of *T*. If we consider that CPL is better represented by *K*, there are two important steps from CPL to Boolean algebra: decidability and factorization. So maybe the algebraic way is not a good way to solve the language-problem equivalence.

8. Countable or uncountable?

Depending on the fact that the domain **F** of *K* is generated by a set of finite, countable[7] or uncountable atomic formulas, **F** is finite, countable or uncountable.

Due to the fact that the notion of isomorphism depends on the notion of cardinality and that the notion of equivalence, whether it is modern and postmodern, depends on the notion of isomorphism, it is not possible to consider that two versions of *K* of different cardinalities are equivalent. This sounds reasonable for the distinction between finite and infinite, since a finite version of *K* differs from an infinite version in many aspects, for example the factorization of a finite version as a finite domain. But this seems artificial for the distinction within the infinite, between countable and uncountable.

The reason why is due to one important feature of the CPL that we have not discussed yet: *substitution* or *structurality* to use the terminology of the famous paper by Łoś and Suszko [9]. It is easier to explain what happens if we consider tautologies rather than a consequence relation. So let us consider the structure $T = \langle \mathbf{F}; \mathbf{T} \rangle$. Following Łoś and Suszko's structuralist approach, a substitution is any endomorphism ε of **F**, and the substitution theorem is expressed by

For every $a \in \mathbf{T}$ *and every* ε, $\varepsilon(a) \in \mathbf{T}$

Given a formula a, the *schema* of *a*, *sch*(*a*), is the set that we get by taking all the formulas of type ε(a), for any endomorphism ε. A *schema*

[7] We use the word countable here by opposition with finite and uncountable.

of formula A is a set of formulas such that there is a formula *a* and *sch*(*a*) = *A*.

The set of schemas of formulas is always countable whether the set of formulas is countable or uncountable and what is important in *T* are schemas of formulas due to structurality. We don't have more tautologies in the case where the cardinality domain of *T* is an inaccesible cardinal since tautologies are viewed as schemas of formula rather than as formulas.

9. Modernism or postmodernism?

There are many CPL-structures, and it is not easy to find a good concept of equivalence that gather them into a whole. The standard concept of equivalence seems inadequate. It can perhaps be adapted, but probably artificially, to solve the problem. At this stage we don't know if the postmodern concept of equivalence is a good one. It has to be tested in fact in the whole universe of logical structures and also outside of it, in the whole universe of mathematical structures, if we want to have a uniform theory. At least there is a problem that it doesn't solve, the question of equivalence between CPL-structures of different cardinalities.

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