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LOGIC AND THE RELATIVITY PRINCIPLE[1]

Abstract. *The logical structure of relativity-like theories is discussed. Some difficulties, arising herein, are pointed.*

My aim is to show how each scientific theory from a fairly wide class can be likened in its logical structure to special relativity. I don't assert that it is interesting to realize such an operation whenever it is possible. However, I give two instances, in which the result applying of the operation provokes some promising stumpers.

1. Logical content of special relativity

The following languages under consideration are all first-order (without identity), if it isn't stipulated for the contrary.

Let *U* be an axiomatic system in the language

$$S = (g_1, \dots ; G_1, \dots , R, \check{R}),$$

where R, Ř are one-place predicate symbols and among the predicate symbols G_1, … there can be the identity symbol ≈. Then, let $\mathring{U}$ be the system obtained from *U* by the substitution of $\breve{g}_1, \dots ; \breve{G}_1, \dots , \check{R}, R$ for, respectively, $g_1, \dots; G_1, \dots , R, \check{R}$ (of course, substituted symbols have suitable arities). The language of $\mathring{U}$ is Љ,

$$\check{S} = (\breve{g}_1, \dots; \breve{G}_1, \dots , \check{R}, R).$$

I will say that axiomatic system *W* in the joint language

$$\Sigma = (g_1, \breve{g}_1, \dots; G_1, \breve{G}_1, \dots , R, \check{R})$$

is $(U, \mathring{U})$-*symmetric*, if $W = \mathrm{Th}\,(U \cup \mathring{U})$, where: U, $\mathring{U}$ are axiomatic systems (in languages S, Љ respectively) of the above-mentioned kind; Th denotes the deductive closure of a set $U \cup \mathring{U}$ of formulas in first-order logic (without identity).

Let *T* be an axiomatic system in the language

$$\sigma = (g_1, \dots; G_1, \dots).$$

Then I say that axiomatic system *W* in the language Σ is $(U, \mathring{U})$-symmetric *over T* if *W* is $(U, \mathring{U})$-symmetric and *U* is an extension of *T* by some defining axiom for R and by the following axiom connecting R and Ř (σ includes neither R nor Ř):

$$\forall x(R(x) \leftrightarrow \neg\check{R}(x)).$$

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Every axiomatic system (U, $\mathring{U}$)-symmetric over T is also ($\mathring{U}$, U)-symmetric over $\check{T}$.

Let me introduce another notion. Let T be such as above. Then I say that axiomatic system W in the language Σ is *weak* (U, $\mathring{U}$)-symmetric over T, if W is (U, $\mathring{U}$)-symmetric and U is an extension of T by some defining axiom for R and by the following axiom connecting R and Ř (σ includes neither R nor Ř):

$$\forall x(\mathrm{R}(x) \rightarrow \neg\check{\mathrm{R}}(x)).$$

Every axiomatic system weak (U, $\mathring{U}$)-symmetric over T is also weak ($\mathring{U}$, U)-symmetric over $\check{T}$.

Now I must make two remarks to motivate the subsequent formulation of the 'logical content of special relativity'.

(1) As it is known [1, p. 112 - 115], *principle of uniform translational motion relativity* reads as follows: under identical initial conditions all physical processes proceed identically in all *inertial* frames (of reference). That, however, doesn't mean that some *given* process is identically observed in all these frames. Au contraire, if, for example, point M_1 is *at rest* relative to some inertial frame, say α, then M_1 *moves* relative to another inertial frame, say β.[2] But if initial conditions of a movement of another point M_2 relative to β are identical with ones of the movement of point M_1 relative to α, then both of these movements will be identical also. In other words, the *equations* of physics have *the same form* in both frames α and β (or, as the saying goes, are *invariant* under the transition from frame α to frame β), but the *results of concrete measurements of concrete* events must, generally speaking, *change* under this transition.

(2) It is also known [*ibid*., 168 - 190], that in special relativity 'simultaneity' is an instrumentally defined notion. This means that any physically meaningful judgement about the simultaneity of two concrete events is an outcome of a certain measuring procedure. But then, by force of (1), this procedure can be reproduced in every inertial frame, on the one hand, and, on the other hand, the concrete results of its execution must, generally speaking, depend on that concrete inertial frame, in which the procedure is executed. Actually, in relativistic physics the considered procedure is defined as follows. If it is realized in frame α for measuring of spatially separated events E_1 and E_2, and if an analogous procedure for measuring of *the same* events is reproduced in frame β, then the results of these measurements satisfy the following two conditionals. If E_1 and E_2 are simultaneous according to the outcome of the measurement in α, then they are not simultaneous accord-

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2 It is supposed that inertial frames α and β move relative to one another with the velocity of v, $0 < v < c$.

ing to the outcome of the measurement in β. If E_1 and E_2 are simultaneous according to the outcome of the measurement in β, then they are not simultaneous according to the outcome of the measurement in α.

Let me express (1) and (2) in other form. Let us pick out that fragment of physics, which is necessary for the said definition of simultaneity. This fragment, being referred to inertial frame α, can always be considered as some consistent axiomatic system T of the forementioned kind.[3] Let us extend T, explicitly defining a new predicate symbol R so as to have possibility to interpret R(x) as the predicate 'if x is a couple of spatially separated (in frame of reference α) events, then x is a couple of simultaneous (in frame of reference α) ones'. Then over again let us proceed this process of extending, introducing another new predicate symbol Ř by the axiom

$$\forall x(\mathrm{R}(x) \rightarrow \neg\check{\mathrm{R}}(x)).$$

Let U be the axiomatic system obtained as the result of these two extensions of T.

Now let us obtain $\mathring{U}$ corresponding to U in the above manner. According to (1) axiomatic system $\mathring{U}$ can be considered as representation (relative to β) of the same fragment of physics, whose representation (relative to α) is axiomatic system U. Herein, obviously, Ř plays the same part in $\mathring{U}$ as R does in U. In other words, Ř(x) can be interpreted as the predicate 'if x is a couple of spatially separated (in frame of reference β) events, then x is a couple of simultaneous (in frame of reference β) ones'. According to (2) systems U and $\mathring{U}$ can be joined in a weak (U, $\mathring{U}$)-symmetric over T axiomatic system $W = \mathrm{Th}\ (U \cup \mathring{U})$ in such a way that W will be a *conservative* extension of T.

The logical content of special relativity is the claim that: physics contains a fragment represented by axiomatic system W; W is weak (U, $\mathring{U}$)-symmetric over T; W is a conservative extension of T; T is a consistent axiomatic system; T corresponds to a fragment of physics also.

2. Generalized relativity principle

Well, the logical content of special relativity is nothing else than a metasentence about two concrete (considered a moment ago) axiomatic systems T and W. This metasentence can be considered as a result of the substitution of concrete T and W for, respectively, metavariables X and Y in the metapredicate 'X is a consistent axiomatic system, Y is a conservative extension of X, and Y is weak (U, $\mathring{U}$)-symmetric over X'. I

[3] For example, T can be obtained by formalizing the chosen fragment of physics within *ZF* and subsequent treatment of the logical identity in the obtained formalism as a signature symbol.

call this metapredicate the *narrow generalized principle of weak relativity* and denote it by $\boldsymbol{G}(X, Y)$.

Let us consider another metapredicate 'X is a consistent axiomatic system, Y is a conservative extension of X, and Y is $(U, \mathring{U})$-symmetric over X'. I call it the narrow generalized principle of *strong* relativity and denote it by $\boldsymbol{G}^{\circ}(X, Y)$. Obviously, for any X, Y implication $\boldsymbol{G}^{\circ}(X, Y) \to \boldsymbol{G}(X,$ *Э*$)$, where *Э* is the result of the substitution of axiom $\forall x(\mathrm{R}(x) \to \neg\check{\mathrm{R}}(x))$ for axiom $\forall x(\mathrm{R}(x) \leftrightarrow \neg\check{\mathrm{R}}(x))$ in Y, is true. Hence sufficient conditions for the satisfiability of the narrow generalized principle of *strong* relativity are also sufficient conditions for the satisfiability of the narrow generalized principle of *weak* relativity. In this connection I indicate the following theorem.

Theorem 1. If X is a consistent axiomatic system and Y is $(U, \mathring{U})$-symmetric over X, then Y is a conservative extension of X if the defining axiom for R in U is such that sentences $\exists x\, \neg\mathrm{R}(x)$ and $\exists x\, \mathrm{R}(x)$ are provable in U.[4]

We have considered two versions (weak and strong) of the narrow generalized principle of relativity. Let us consider also two analogues ones of a less restrictive generalized principle of relativity. I call the metapredicate 'X is a consistent axiomatic system, Y is a consistent extension of X, and Y is weak $(U, \mathring{U})$-symmetric over X' the *broad* generalized principle of *weak* relativity. And I call the metapredicate 'X is a consistent axiomatic system, Y is a consistent extension of X, and Y is $(U, \mathring{U})$-symmetric over X' the *broad* generalized principle of *strong* relativity. Interrelations between these two new versions are analogues to old ones. The following theorem takes place here.

Theorem 2. If X is a consistent axiomatic system and Y is $(U, \mathring{U})$-symmetric over X, then Y is a consistent extension of X if the defining axiom for R in U is such that sentences $\forall x\, \neg\mathrm{R}(x)$ and $\forall x\, \mathrm{R}(x)$ are not provable in U.[5]

3. Commentary

The above motivates the following definition. I say that axiomatic system Y is *like in logical structure to special relativity* if Y has some consistent subsystem X such that X and Y fulfil the narrow or broad generalized principle of weak or strong relativity. But then the above-mentioned theorems say, virtually, how and when arbitrary scientific theory X can be likened to special relativity, i.e., transformed into some theory Y which is like in logical composition to special relativity. But,

[4] Cp.: [2], p. 75-77.
[5] Cp.: [3], p. 52-57.

naturally, they do not say that it is worth our effort to do so whenever it is possible. Everything depends on concrete circumstances of non-formal character. I will adduce two examples so as to give the reader a hint what casuses can herein happen.

Elementary (Peano) arithmetic *N* can be considered as an axiomatic system in a first-order language *without* identity if the symbol of identity ≈ is regarded as a signature symbol. Let A(*x*) be some arithmetic formula weakly defining a recursively enumerable non-recursive set **R** in *N* (*N* is supposed to be consistent). Suppose, furthermore, that sentences ∃*x* ¬A(*x*) and ∃*x* A(*x*) are provable in *N*.[6] Let us extend arithmetic *N* by two new axioms:

(i) $\forall x\,(\mathrm{R}(x) \leftrightarrow \mathrm{A}(x))$;

(ii) $\forall x\,(\mathrm{R}(x) \leftrightarrow \neg\check{\mathrm{R}}(x))$.

Let us refer to the obtained system of language

$$S_0 = (+, *, ґ, 0; \approx, \mathrm{R}, \check{\mathrm{R}})$$

as U_0. Having U_0, we can, in the same way as before, obtain system U_0. Obviously, the language of U_0 will be

$$Љ_0 = (\breve{+}, \breve{*}, \breve{'}, \breve{0}; \breve{\approx}, \check{\mathrm{R}}, \mathrm{R}).$$

Let us consider a system W_0 of language

$$\Sigma_0 = (+, \breve{+}, *, \breve{*}, ', \breve{'}, 0, \breve{0}; \approx, \breve{\approx}, \mathrm{R}, \check{\mathrm{R}}),$$

having defined W_0 by

$$W_0 = \mathrm{Th}\,(U_0 \cup U_0).$$

It is clear that W_0 is (U_0, U_0)-symmetric over *N* and satisfies the conditions of theorem 1 *qua Y* when *N* is considered *qua X*. Consequently, *N* and W_0 satisfy the narrow generalized principle of strong relativity:

$$\boldsymbol{G}^\circ(N, W_0).$$

But this fact causes some stumpers.

As I noted before, formula A(*x*) is fixed so that it weakly defines recursively enumerable non-recursive set **R** in *N* and, consequently, in U_0 (for U_0 is a conservative extension of *N*). In the same way one could claim that formula Ă(*x*) which is obtained from A(*x*) in the above-mentioned transition from U_0 to U_0 weakly defines some recursively enumerable non-recursive set, say $\check{\mathbf{R}}$, in $\check{N}$ and, consequently, in

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[6] Number n-place relation Q is called weakly definable in system *S* of signature $\Omega \supseteq \sigma$, where σ is the signature of Peano arithmetic, if there exists formula B of signature Ω with free variables $x_1, \ldots, x_n$ such that for numbers $k_l, \ldots, k_n$ and their numerals $\boldsymbol{k_l}, \ldots, \boldsymbol{k_n}$ $(k_l, \ldots, k_n) \in \mathrm{Q} \Leftrightarrow \mathrm{B}(\boldsymbol{k_l}, \ldots, \boldsymbol{k_n})$ is provable in *S*.

If Peano arithmetic is consistent and system *S* is a conservative extension of this arithmetic, then a number relation is weakly definable in *S*, iff the relation is recursively enumerable.

U_0 (for U_0 is a conservative extension of $\check{N}$). What - recursively enumerable or not - sets do weakly define formulae A(x) and Ă(x) in system W_0? The question has no definite answer and, consequently, is meaningless. Indeed, on the one hand, W_0 extends conservatively systems N and $\check{N}$, and these systems (and formulae A(x) and Ă(x)) play completely symmetric parts in W_0. On the other hand, according to axioms (i), (ii) sentence $\forall x$ (A(x) $\leftrightarrow$ ¬Ă(x)) is derivable in W_0. In the issue, *if* it is N that is accounted to be Peano arithmetic, we must consider that set **R** *is* recursively enumerable and set **Ř** is *not* recursively enumerable. Au contraire, *if* it is $\check{N}$ that is accounted to be Peano arithmetic, we must consider that set **R** is *not* recursively enumerable and set **Ř** *is* recursively enumerable. To ask herein, which of these systems N and $\check{N}$ is a genuine arithmetic and which is not is as absurdly as to ask in special relativity, which of inertial frames α and β is a correct frame of reference and which is not. The hell of it is not that only one of systems N and $\check{N}$ describes a ‘truly’ arithmetic — both they equally can be accounted to be descriptions of arithmetic. The hell of it is that within W_0 the ‘absolute’ notion ‘to be a recursively enumerable set’ does not make sense, but the ‘relativistic’ notion ‘to be a recursively enumerable set *relative to N* (or $\check{N}$)’ does. Analogy with the relativistic notion of simultaneity is obvious.

In this connection another question arises: how to be with Church’s thesis? Two approaches are possible here.

Firstly, we can consider that in the context of W_0 Church’s thesis amounts to the biconditional: set **R** is effectively enumerable iff set **Ř** is not effectively enumerable. That doesn’t entail any new interesting effects for the philosophy of mathematics and, consequently, imparts the status of a formal prestidigitation to the fact of relativity of recursive enumerability.

Secondly, we can attempt to relativize an effectiveness in itself as an *empirical* possibility to run (in principle) a program on a physical model of an abstract universal Turing machine. A possible success of such an attempt may have an essential influence on the philosophy of mathematics. In its turn, the essential first step on the road to the success must, obviously, presuppose a successful search for determinate physical processes such that any computer working according to the common physical effects cannot simulate them in principle. I don’t know whether somebody or other is engaged in such research.

Let us consider another example. It is generally assumed that in so-called ‘constitutional states’ a man can be officially proclaimed to be a criminal if only the jury brought in the verdict of his guilty and only on a concrete charge. This means that any such adjudication presupposes a concrete statement of offence and the ascertainment of the fact of this statement's applicability to the given person (to the indictee).

What is a concrete statement of offence? However legalists treat this notion, from a logical point of view it is, at any rate, some one-place predicate, say R(x), considered as the abbreviation for 'x is guilty of thus-and-so'. Herein it is supposed that this predicate is *definable* in some — let me call it jural — *theory*. The last can comprise civil and criminal codes, bylaws, delegated legislation etc. I'll forgo further elaboration of its structure and foundation here. Instead of it, I artlessly suppose that — from a logical point of view again — the jural theory can always be represented by axiomatic system T of the above-mentioned kind. Thus I suppose that law court deals with axiomatic system D which is the extension of T *via* defining axiom for R:

$$D = \mathrm{Th}(T \cup \{\forall x\, (\mathrm{R}(x) \leftrightarrow \mathrm{F}(x))\}),$$

where F(x) is a definiens in T for R(x).

It is worthy of note that if law court doesn't degenerate into a farce, then defining axiom $\forall x\, (\mathrm{R}(x) \leftrightarrow \mathrm{F}(x))$ defines statement of offence R(x) so that the following condition (*non-degeneration condition*) is satisfied: sentences $\forall x\, \mathrm{R}(x)$, $\forall x\, \neg\mathrm{R}(x)$ aren't provable in D.

The models of axiomatic system D having arbitrary communities as their universes — let ***M*** will be the class of all such ones — can be considered as *possible* (*legit*, *legally allowable*) *explanations* (*glosses*) of terms used in the lawyers' dialect. Accordingly, a choice of a concrete model from ***M*** can be considered as a choice of such a concrete gloss.

The process develops according to the following schema. Concrete model $M \in$ ***M*** containing the given indictee in the membership of its universe is choiced. Law court ascertains — amiss or not — whether the indictee belongs to subset **R** (of M's universe $|M|$) corresponding to predicate R(x). If the law court decides that the indictee belongs to **R**, then he is proclaimed to be a criminal. If it decides that the indictee doesn't belong to **R**, then he is proclaimed to be an innocent. It cannot be emphasized enough that adjudgement depends substantively on the above-mentioned choice of the gloss M. The responsibility for this choice rests with the law court, since any other model, say M', from ***M*** such that the indictee belong to $|M'|$ is as legally allowable as M.

If we extend system D *via* axiom $\forall x(\mathrm{R}(x) \leftrightarrow \neg\check{\mathrm{R}}(x))$, designate the extended system by U, obtain, in the above mentioned way, system $\mathring{U}$ first and (U, $\mathring{U}$)-symmetric over T system W after, then we see that the non-degeneration condition coincides (within the accuracy of notation) with the conditions of theorem 2. This means that system W is consistent and, consequently, always has models. Obviously, the class of these models includes all models from ***M*** whose universes contain the indictee. Let us consider such an arbitrary model *K*. The reduct of this model to the signature of system *D* gives model *L* which, obviously, belongs to class ***M*** and, consequently, is a legally allowable gloss. The reduct of this model to the signature of system *Ď* gives a model *L*r which, obviously, also belongs to class ***M*** and, consequently, also is a legally allowable gloss. Let **I** be the subset of set |*K*| corresponding to predicate R(*x*) in model *L*. Let **I** be the subset of set |*K*| corresponding to predicate R(*x*) in model *L*r (in model *K* subset I corresponds to predicate Ř(*x*)). Then by virtue of axiom $\forall x(R(x) \leftrightarrow \neg \check{R}(x))$ we have for any *x* from |*K*|: $x \in$ I iff $x \notin$ I. Therefore the fact that system *W* has a model at all means in part the following: law court always has a choice between two legit glosses such that the indictee is a criminal according to one gloss but he is innocent according to the other.

Can this circumstance serve as a permanent pretext for taking an appeal? If it can't, then why it is so? If it can, then how must an appeals instance react to it?

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