ŁUKASIEWICZ LOGICS
AND
PRIME NUMBERS

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Introduction

"…*the philosophical significance of the* [*many-valued*] *systems of logic treated here might be at least as great as the significance on non-Euclidean systems of geometry*."

J. Łukasiewicz, 1930.

"…*there is no apparent reason why one number is prime and another not. To the contrary, upon looking at these numbers one has the feeling of being in the presence of one of the inexplicable secrets of creation*."

D. Zagier, 1977.

The present book is a substantially rewritten English version of its Russian predecessor ([Karpenko, 2000]).[1]

The title of the book may appear somewhat strange since, at first glance, what can logic and prime numbers have in common? Nevertheless, for a certain class of finite-valued logics such commonalties do exist – and this fact has a number of significant repercussions. Is there, however, any link between the doctrine of logical fatalism and prime numbers?

Jan Łukasiewicz (1878-1956) was a prominent representative of the Lvov-Warsaw School (see [Woleński, 1989] for details) and the originator of mathematical investigation of logic within that school. His criticism of Aristotle's fatalistic argument laid the ground for the historically first non-classical, three-valued, logic. Its properties proved to be – for Łukasiewicz's time – somewhat shocking; its subsequent generalizations for an arbitrary finite and – further still – the infinite cases showed that the modeling of the infinite and the finite on the basis of Łukasiewicz many-valued logics yields results that justify the claim that, by the end of the twentieth century, there have taken shape and are now rapidly growing two distinct and significant trends in the contemporary symbolic logic: Łukasiewicz infinite-valued logic $\mathbf{Ł}_\infty$ (see [Cignoli, D'Ottaviano and Mundici, 2000]) and Łukasiewicz finite-valued logics $\mathbf{Ł}_{\mathbf{n+1}}$ – the logics discussed in this book. While in the

[1] See A. Adamatzky's English extensive review of this book [Adamatzky, 2004].

former case the beauty of the subject arises out of consideration of different (but equivalent) algebraic structures serving as counterparts of the logic as well as out of its various applications; in the latter, we enter the mystical world of prime numbers, the world that proves to be connected with the functional properties of $\mathbf{Ł}_{n+1}$.

The book consists of three parts, dealing with, respectively, (1) Łukasiewicz finite-valued logics $\mathbf{Ł}_{n+1}$; (2) their link with prime numbers; and, lastly, (3) the numeric tables illustrating the link described in part (2).

Chapter I is an elementary introduction to the two-valued classical propositional logic $\mathbf{C}_2$. It is worth noticing that Łukasiewicz two-valued logic $\mathbf{Ł}_2$ is nothing else than $\mathbf{C}_2$. It means that all Łukasiewicz many-valued logics are generalizations of $\mathbf{C}_2$. Chapter II describes the origin and development of Łukasiewicz three-valued logic $\mathbf{Ł}_3$ and indicates the connection between $\mathbf{Ł}_3$ and the problem of logical fatalism. Some surprising and unexpected properties – such as the failure of "the laws" of excluded middle and non-contradiction – of $\mathbf{Ł}_3$ are also considered there; that consideration makes apparent that, as soon as we introduce some novelties into the classical logic, there arises a thorny problem of what interpretations of the logical connectives and of the truth-values themselves are intuitively acceptable. (This problem, in turn, leads to the problem of what is a logical system – all the more so, given the at first glance surprising fact that $\mathbf{Ł}_3$, as well as any other $\mathbf{Ł}_{n+1}$, can be axiomatically presented as a restriction of a Hilbert-style axiomatization of $\mathbf{C}_2$ and also as an extension of a Hilbert-style axiomatization of $\mathbf{C}_2$.)

In Chapter III we consider some properties of $\mathbf{Ł}_{n+1}$, including degrees of cardinal completeness of $\mathbf{Ł}_{n+1}$'s (first studied by A. Tarski in 1930) – the property that allowed us the first glimpse of a connection between $\mathbf{Ł}_{n+1}$'s and prime numbers. Towards the end of Chapter III, we propose an interpretation of $\mathbf{Ł}_{n+1}$ through Boolean algebras.

In our view, neither the axiomatic nor the algebraic (nor, for that matter, any other semantic) approach can bring out the uniqueness and peculiarity of Łukasiewicz finite-valued logics $\mathbf{Ł}_{n+1}$. All these approaches we call external, as opposed to the approach considering $\mathbf{Ł}_{n+1}$'s as functional systems. We believe that only the latter approach can help us decipher the essence of $\mathbf{Ł}_{n+1}$'s. It was exactly this approach that allowed to discover that functional properties of $\mathbf{Ł}_{n+1}$ are highly unusual. V.K. Finn was the first to note this in his brief paper "On classes of functions that corresponded to the *n*-valued logics of

J. Łukasiewicz" ([Finn, 1970]). One repercussion of Finn's work is that the set of functions of the logic $\mathbf{Ł_{n+1}}$ is functionally precomplete if and only if *n* is a prime number. Finn's result is discussed in Chapter IV, which is *crucial for our consideration* because it provides a bridge between the first and the second parts of the book. (It should be noted that Finn's result – which was later independently re-discovered – is both the foundation of and the primary inspiration for the writing of the present book.)

The Finn's led to an algorithm mapping an arbitrary natural number to a prime number with the help of the Euler's totient function $\varphi(n)$, thus inducing a partition of the set of natural numbers into classes of equivalence; each of thus obtained classes can be represented by a rooted tree of natural numbers with a prime root. That algorithm, in turn, led to an algorithm based on some properties of the inverse Euler's totient function $\varphi^{-1}(m)$ mapping an arbitrary prime number *p* to an equivalence class equivalence of natural numbers. Chapter V contains thus obtained graphs for the first 25 prime numbers as well as the canceled rooted trees for prime numbers from 101 (No. 26) to 541 (No. 100). Thus, each prime number is given a structure, which proves to be an algebraic structure of *p*-Abelian groups.

Some further investigations led to the construction of the finite-valued logics $\mathbf{K_{n+1}}$ that have tautologies if and only if *n* is a prime number ($\mathbf{K_{n+1}}$ are described in Chapter VI). The above statement can be viewed as a purely logical definition of prime numbers. $\mathbf{K_{n+1}}$ happen to have the same functional properties as $\mathbf{Ł_{n+1}}$ whenever *n* is a prime number. This provided the basis for constructing the Sheffer stroke operator for prime numbers. (In this construction, we use formulas with 648 042 744 959 occurrences of the Sheffer stroke.) It is interesting that a combination of logics for prime numbers helps discover *a law of generation of classes of prime numbers*. As a result, we get a partition of the set of prime numbers into equivalence classes that are induced by algebraic-logical properties of Łukasiewicz implication; all prime numbers can be generated in such a way.

Finally, in Chapter VII we give what we consider to be an ultimate answer to the question of what is a Łukasiewicz many-valued logic. Its nature is purely number-theoretical; this is why it proves possible to characterize, in terms of Łukasiewicz logical matrices, such subsets of the set of natural numbers as prime numbers, powers of primes, odd numbers, and – what proved to be the most difficult task – even numbers. (In that last case, we also try to establishing a link with

Goldbach's conjecture concerning the representation of every even number by the sum of two prime numbers.)

The third part of the book is made up of the numerical tables never previously published. Table 1 contains the values of the cardinal degrees of completeness for *n*-valued Łukasiewicz logics ($n \leq 1000$). (Some natural numbers, in this respect, happen to form a special "elite".) In table 2 the cardinality values of rooted trees and of canceled rooted trees are given for $p \leq 1000$. Table 3 in this book gives the values of function $i(p)$, for $p \leq 1000$, which partitions prime numbers into equivalence classes.

In algebraic terms, in this book we investigate the clones of finite MV-chains.

Concluding remarks contain some metaphysical reflections on extensions of *pure logic*, on prime numbers, and on fatalism and continuality, as well as on the possible connections of these themes with Łukasiewicz logics.

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Alexander S. Karpenko,

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