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THE CLASSIFICATION OF PROPOSITIONAL CALCULI
(Finite Boolean Latticies of Implicational Logics)

Abstract. *We discuss Smirnov's problem of finding a common background for classifyinn of implicational logics. We formulate and solve the problem of extending, in an appropriate way, an implicational fragment* $\mathbf{H}_{\to}$ *of the inuitionistic propositional logic to an implicational fragment* $\mathbf{TV}_{\to}$ *of the classical propositional logic. As a result we obtain logical constructions having the form of Boolean lattices whose elements are implicational logics. In this way, whole classes of new logics can be obtained. We also consider the transition from implicational logics to full logics. On the base of the lattices constructed, we formulate the main classification principles for propositional logics.*

1. Introduction

The classification problem for logical calculi was posed in 1972 by V.A.Smirnov [36]. The classification of singular sequential calculi was also suggested there, which gives rise, in turn, to the classification for rules of introduction and elimination of logical connectives.

It was the first time that structural rules were used for the classification of logical calculi. Much later, the same idea independently arose and was widely used by several researchers (see Belnap [3], Došen [8], [9], Ono [27], Wansing [47]).

Smirnov comes to the structural rules from comparing different concepts of formal inference. The deduction theorem takes different forms, when formal inference varies in the structure. This fact allows him to classify implicational logics according to the form which the deduction theorem takes.

One more classification of implicational logics based only on structural rules was suggested by Smirnov in [37], where the correspondence between some implicative formulas and structural rules was established. Smirnov pays attention to the very important problem concerning the suggested classifications: in the first case, the deduction theorem has the same form for $\mathbf{H}_{\to}$ and $\mathbf{TV}_{\to}$, the implicational fragments of intuitionistic and classical logics, and there is then no distinction between the logics $\mathbf{H}_{\to}$ and $\mathbf{TV}_{\to}$. In the second case, we can not point out a structurul rule providing the transition from $\mathbf{H}_{\to}$ to $\mathbf{TV}_{\to}$. This transition is usually realized by adding Peirce's law

$$\mathbf{P}.\ ((p \to q) \to p) \to p.$$

However, there is no structural rule corresponding to this formula.

There is a quite different approach to the classification of implicational logics, which uses the properties of basic (initial) combinators **I**, **B**, **C**, **W**, **K,** and **S** introduced by first M.Schonfinkel [34] and subsequently by H.Curry (see [6]). We may consider these combinators as simple operators of reordering brackets and canceling and/or duplicating terms they are applied to:

$\mathbf{I}x = x$.	$\mathbf{W}xy = xyy$,
$\mathbf{B}xyz = \mathrm{x(yz)}$,	$\mathbf{K}xy = x$,
$\mathbf{C}xyz = \mathrm{xzy}$,	$\mathbf{S}xyz = xz(yz)$,

where x, y, and z are arbitrary terms.

Further combinators are generated from the initial ones, for example, $\mathbf{B}'xyz = x(zy)$ (= **CB**), $\mathbf{I}'\,xy = yx$ (= **CI**). It turns out that the following sets of combinators {**B**, **C**, **W**, **K**}, {**B′**, **W**, **K**}, and {**S**, **K**} are equivalent. For the latter (and so for the others) combinatorial completeness is established, which means that all possible combinators are generated from combinators occuring in one of these sets. As is known there is an isomorphic correspondence (the so-called Curry-Howard isomorphism) between combinators and implicative formulas [6, ch. 9E]. The main consequence of this isomorphism is that the complete set of initial combinators defines the intuitionistic implication $\mathbf{H}_{\rightarrow}$. We can use the Curry-Howard isomorphism to classify implicational logics in terms of combinators and vice-versa [11].

However, this classification also does not include the classical implicational logic $\mathbf{TV}_{\rightarrow}$, because there is no combinator corresponding to Peirce's law or to any non-intuitionistic implicative formula. This explains why one of the main goals of Gabbay and de Queiroz's work [11] was to extend the Curry-Howard isomorphism to $\mathbf{TV}_{\rightarrow}$, i.e., to construct a «combinator» **P** corresponding to Peirce's law, and it was done in a sophisticated way.

So, the following initial problem, which we call Smirnov's problem, lies in front us: to find a common background for the classification of implicational logics covering $\mathbf{TV}_{\rightarrow}$. In addition, we will try we try to extend this classification to other types of logics, first of all to full logics, i.e., to logics with all thebasic logical connectives.

2. The lattice of implicational logics $L(\mathbf{H}_{\rightarrow})$

The problem envisaged in the introduction will be resolved by the presentation of a logical construction which involves all the logics in question. Moreover, applying the simplest operations to the construction presented generates new logics and even infinite classes of logics. As primitive objects for our construction we take the following implicative formulas:

I. $p \to p$

B. $(q \to r) \to ((p \to q) \to (p \to r))$

C. $(p \to (q \to r)) \to (q \to (p \to r))$

W. $(p \to (p \to q)) \to (p \to q)$

$\mathbf{K_1}$. $(p \to q) \to (r \to (p \to q))$.

The operations are the usual inference rules:

R1. *Modus ponens*: $A \to B$ and A imply B.
R2. *Substitution* for propositional variables.

For example, the formula $\mathbf{K_1}$ is a result of simultaneous substitution in **K**:

K. $(p \to (q \to p))$,

where $p \to q$ is substituted for p and r for q, i.e. $p/p \to q$ and q/r.

We denote by |- A the provability of a formula A and write down the proofs in a way suggested by J.Łukasiewicz [23]. Every thesis proved will be numbered and preceded by a proof line, which consists of two parts separated by an asterisk. For instance, let us consider the following proof.

Proposition 1. W, K |- I.

1. **W.**
2. **K.**

1 q/p $*$ 2 q/p - 3,

3. $p \to p$ (= **I**).

Here, the first part of the proof line indicates that p is substituted for q in thesis 1, the second part indicates the substitution in thesis 2. Thus, applying modus ponens to the results of substitution we prove thesis 3.

Note that the set of implicative formulas chosen as primitive objects should be independent – the fundamental requirement imposed on sets of primitive objects.

Theorem 1. *The set of formulas* ***I, B, C, W,*** $\boldsymbol{K_1}$ *provides an independent axiomatization of* $\boldsymbol{H_{\to}}$.

The proof consists of two parts:

(i) the independence proof for formulas **I, B, C, W,** $\mathbf{K_1}$;
(ii) the proof that **I, B, C, W,** $\mathbf{K_1}$ axiomatize $\mathbf{H_{\to}}$**.**

(i) We use the matrix method. All the matrices involved are normal in the sense of Łukasiewicz-Tarski [23], i.e., they verify the modus ponens.

Matrix 1 [38]

$\to$	0	1	2		
0	2	2	2		
1	0	0	2	verifies	falsifies
*2	0	0	2	**B,C,W,K$_1$**	**I** (p=1).

Matrix 2 [1, p.85]

$\to$	0	1	2		
0	2	2	2		
1	2	2	2	verifies	falsifies
*2	0	2	2	**I,C,W,K$_1$**	**B** (p=2, q=1, r=0).

Matrix 3 [38]

$\to$	0	1	2		
0	2	2	2		
1	0	2	2	verifies	falsifies
*2	0	0	2	**I,B,W,K$_1$**	**C** (p=2, q=1, r=1).

Matrix 4 [20]

$\to$	0	1	2		
0	2	2	2		
1	1	2	2	verifies	falsifies
*2	0	1	2	**I,B,C,K$_1$**	**W** (p=1, q=0).

Matrix 5 [38]

$\to$	0	1	2		
0	2	2	2		
*1	0	1	2	verifies	falsifies
*2	0	0	2	**I,B,C,W**	**K$_1$** (p=0, q=0, r=1)

(ii). It is enough to show that **I, B, C, W, K$_1$** |- **K**.

Proposition 2. **I**, **C**, $\mathbf{K_1}$ |- **K** [36 , p.61].

1. **I**.
2. **C**.
3. $\mathbf{K_1}$.
   3 q/p, r/q $*$ 1 - 4,
4. $q \rightarrow (p \rightarrow p)$.
   2 p/q, q/p, r/p $*$ 4 - 5,
5. $p \rightarrow (q \rightarrow p)$ (=**K**).

Theorem 1 is thus proved..

Now in virtue of the independence of the set {**I, B, C, W,** $\mathbf{K_1}$}, we can build the following *logical construction*. It is wellknown that the family of all subsets of a set forms a Boolean lattice w.r.t. inclusion. Considering the family of subsets of the above set of formulas we obtain our logical construction denoted by *L*(**H**) a Boolean lattice with 32 (=2^5) elements and $\mathbf{H}_{\rightarrow}$ as its unit. For simplicity of drawing we take the logic **IB** to be the zero of the lattice. As a result we have the following eight-element lattice:

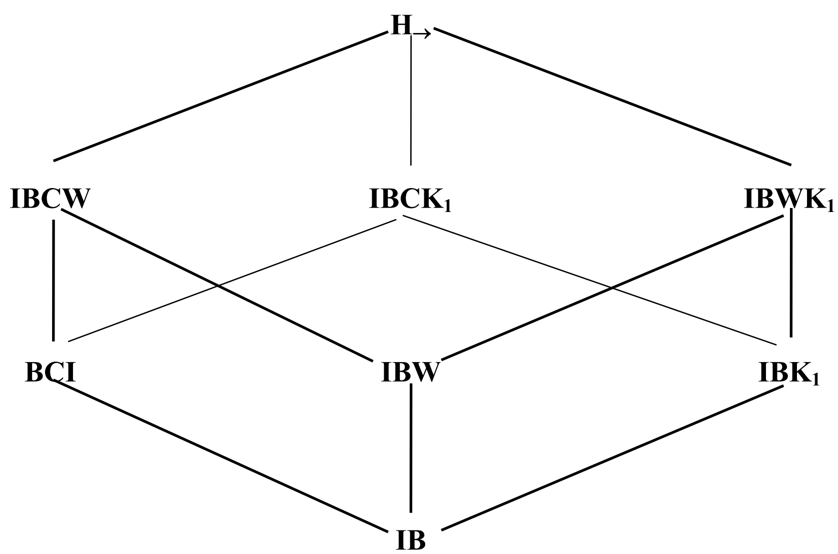

Fig. 1

The logic **IBCW** is the Church weak positive implication $\mathbf{R}_{\rightarrow}$ [5]. In view of Propositions 1 and 2 and the fact that $\mathbf{K_1}$ is a substitutional instance of **K**, we have $\mathbf{IBCK_1} \equiv \mathbf{BCK}$. A.Prior [29, p.316] wrote that **BCI** and **BCK** were introduced by C.A.Meredith in 1956. However, it should be noted that **BCK** as a logical system was isolated as early as in 1934 by A.Tarski [40]. We note also that H.Curry [6] proved the deduction theorem for **IB**.

3. The lattice of implicational logics $L(\mathbf{TV}_{\rightarrow})$

We deal here with a problem similar to that envisaged in the end of Section 1: Does there exist a formula **X** such that the set of formulas **I, B, C, W, $\mathbf{K_1}$, X** provides an independent axiomatization of $\mathbf{TV}_{\rightarrow}$ [14]?

Recall that, in virtue of the Tarski-Bernays theorem (see [33]), the classical implicational logic $\mathbf{TV}_{\rightarrow}$ is axiomatized by formulas **B', K, P** with modus ponens and substitution rules, where

$$\mathbf{B'}.\ (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)).$$

As was noted above, $\mathbf{TV}_{\rightarrow}$ can be obtained by addition of **P** to $\mathbf{H}_{\rightarrow}$. But Peirce's law is not a satisfactory candidate for **X**, because the formulas **I, B, C** + **P** already axiomatize $\mathbf{TV}_{\rightarrow}$ **,** i.e., **I B, C, P** |- **W, K,** which means that the set of formulas **I, B, C, W, K, P** is not independent. Since **B, C** ≡ **B', C** it sufficies to prove

Proposition 3. I, B, C, P |-**K** (see [46], [29, p.318], [39]).

Thus, the formula **P** must be weakened. A suitable weakening of **P** was found in November 1992 by the author:

$$\mathbf{X_1}.\ ((p \rightarrow q) \rightarrow ((r \rightarrow r) \rightarrow (p \rightarrow q))) \rightarrow (\mathbf{W_1} \rightarrow \mathbf{P_1}),$$

where $((p \rightarrow q) \rightarrow ((r \rightarrow r) \rightarrow (p \rightarrow q)))$ is a substitutional instance of $\mathbf{K_1}$: $r/r \rightarrow r$; $\mathbf{W_1}$ of **W**: $p/p \rightarrow q$, q/r; and $\mathbf{P_1}$ of **P**: $p/p \rightarrow q$, q/r.

Theorem 2. *The set of formulas **I, B, C, W, $\mathbf{K_1}$ $\mathbf{X_1}$** provides an independent axiomatization of* $\mathbf{TV}_{\rightarrow}$.

(i). We prove the independence of **I, B, C, W, $\mathbf{K_1}$** using the matrices from Theorem 1. The independence of $\mathbf{X_1}$ can be proved by

Matrix 6 (three-valued implication of Heyting [13])

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
*2	0	1	2

verifies **I,B,C,W,$\mathbf{K_1}$** — falsifies $\mathbf{X_1}$ (p=2, q=1, r=0).

(ii). **I, B, C, W, $\mathbf{K_1}$, $\mathbf{X_1}$** is $\mathbf{TV}_{\rightarrow}$:

Proposition 4. I, B, C, W, $\mathbf{K_1}$, $\mathbf{X_1}$ |- P.

$\mathbf{X_1} * \mathbf{K_1}\ r/r \rightarrow r$ - $\mathbf{W}\ p/p \rightarrow q,\ q/r$ - $\mathbf{P_1}$,

$\mathbf{P_1}.\ (((p \rightarrow q) \rightarrow r) \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q).$

Further, see Addition 2 in [16] where it was proved that **I, B, C, P_1** |- **P** (the proof contains 23 numbered formulas).

We have thus constructed a suitable extension of $\mathbf{H}_\to$ of $\mathbf{TV}_\to$. But the formula $\mathbf{X_1}$ can be simplified. Note that $\mathbf{W_1} \to \mathbf{P_1}$ is a substitutional instance of a formula **D**:

D. $((p \to q) \to q) \to ((q \to p) \to p)$.

Consider a formula $\mathbf{X_2}$:

$(p \to ((q \to q) \to p)) \to$ **D**,

where$(p \to ((q \to q) \to p))$ is a substitutional instance of **K**: $q/q \to q$.

Theorem 3. *The set of formulas* ***I, B, C, W,*** $\boldsymbol{K_1}$, $\boldsymbol{X_2}$ *provides an independent axiomatization of* $\boldsymbol{TV}_\to$ (compare [15]).

(i). The independence of **I, C, W,** $\mathbf{K_1}$**,** $\mathbf{X_1}$ is proved by the same matrices as in Theorem 2**.** The independence of a formula **B** is proved by

Matrix 7 [18], [45]:

$\to$	0	1	2	3		
0	3	3	3	3		
1	3	3	2	3		
2	3	1	3	3	verifies	falsifies
*3	0	1	2	3	**I,C,W,**$\mathbf{K_1}$**,**$\mathbf{X_2}$	**B** (p=2, q=0, r=1)

(ii). **I, B, C, W,** $\mathbf{K_1}$**,** $\mathbf{X_2}$ is $\mathbf{TV}_\to$:

Proposition 5. I, C, W, $\mathbf{K_1}$**,** $\mathbf{X_2}$ **|- P.**

1. **I**.
2. **C**.
3. **W**.
4. $\mathbf{K_1}$.
5. $\mathbf{X_2}$.
6. **I**, **C**, $\mathbf{K_1}$ **|- K** (Proposition 2).

5 $q/p \to q * 6\ q/(p \to q) \to (p \to q)$ - 3 - 7,

7. $((p \to q) \to p) \to p$ (= **P**).

The proof of Theorem 3 is complete.

Now we consider the lattice $L(\mathbf{TV}_\to)$ consisting of logics generated by formulas in {**I, B, C, W,** $\mathbf{K_1}$**,** $\mathbf{X_2}$}. For simplicity of drawing we take **BCI** as the zero of the lattice:

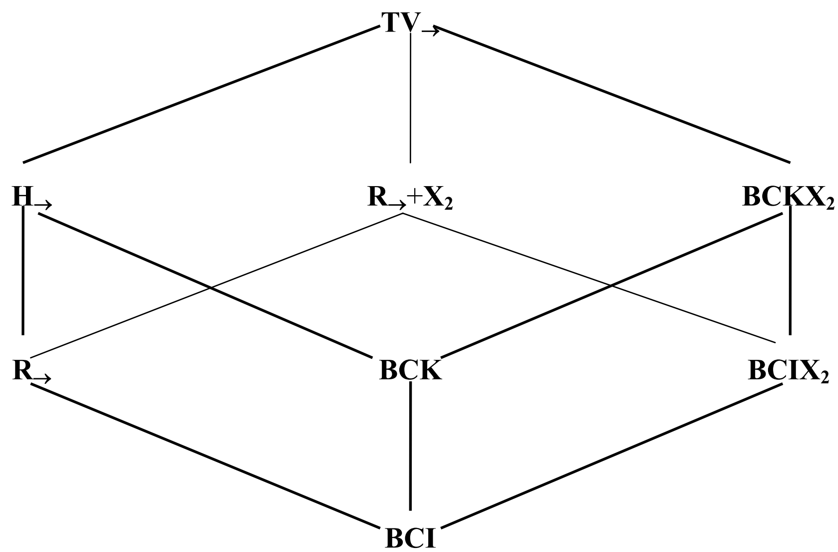

Fig. 2

The logic $\mathbf{BCKX_2}$ is of special interest.

Proposition 6. $\mathbf{BCKX_2 = BCKD}$.

The proof is evident.

BCKD is a fragment of Łukasiewicz's infinite-valued logic $Ł_\omega$ [23] and was studied for the first time by A.Rose and J.Rosser [33] (see also [10]). The following equalities hold

$$\mathbf{BCKD = BCID = B'KD = BKD}.$$

The logic **B'KD** has the following remarkable property. If we add to **B'KD** the linearity law **L**:

$$\mathbf{L}.\ ((p \to q) \to (q \to p)) \to (q \to p),$$

we obtain an implicational fragment $Ł_{\omega\to}$ of $Ł_\omega$ [31], [25].

Now we pay attention to the following important fact: though

$$\mathbf{IBCWK_1X_1 = IBCWK_1X_2 = TV_\to}$$

the logical constructions corresponding to these axiomatizations are different. For example, we have

Proposition 7. $\mathbf{BCKX_1 \neq BCKX_2}$.

Matrix 8

$\to$	0	1	2	3		
0	3	3	3	3		
1	2	3	3	3		
2	2	2	3	3	verifies	falsifies
*3	0	1	2	3	**I,B,C,K$_1$,X$_1$**	**X$_2$** (p=1, q=0).

The formula **X$_1$** is provable in **BCKX$_2$**. Thus, we have different classical versions of the logic **BCK**.

4. The maximal lattice L(TV$_\to$): RM$_\to$ and Ł$_{\omega\to}$ logics

The different **TV$_\to$**-constructions give rize to the question concerning the class of possible formulas **X**$_i$ [16, p.242]. J.Slaney and M.Bunder [35, p.64] posed also the following two problems:

"(1) Is there an infinite number of distinct systems **BCIX$_i$**, **BCKX$_i$** and **BCIWX$_i$**?

(2) Is there a weakest and strongest system **BCIX$_i$**, **BCKX$_i$** and **BCIWX$_i$**?"

In [35], another formula was taken to be an **X**:

$$\mathbf{X_3}.\ ((((p \to q) \to q) \to p) \to r) \to (((((q \to p) \to p) \to q) \to r) \to r).$$[1]

Slaney and Bunder showed that

BCKX$_2$ $\neq$ **BCKX$_3$**

and that **X$_2$** is provable in **BCKX$_3$**.

However, the independence of **I** from **B, C, W, K$_1$**, **X$_3$** was not proved in [35]. Moreover, there is not even a four-element matrix for checking this fact. This is our reason replacing **X$_3$** by the following formula **X$_4$**:

$$\mathbf{X_4}.\ (p \to p) \to \mathbf{X_3}.$$

Theorem 4. *The set of formulas **I, B, C, W, K$_1$**, **X$_4$** provides an independent axiomatization of **TV$_\to$**.*

(i). The independence proof follows the line of Theorem 2.

(ii). **I, B, C, W, K$_1$, X$_4$** is **TV$_\to$**:

[1]This formula is due to [26] in which an independent axiomatization of **RM$_\to$** was proposed with formulas **B', W, I', X$_3$**, where **I'** is $p \to ((p \to q) \to q)$.

Proposition 8. I, B, C, W, K_1, X_4 |- P.

1. $(p \to \mathrm{p})$ (= **I**).
2. $(q \to r) \to ((p \to q) \to (p \to r))$ (= **B**).
3. $(p \to (q \to r)) \to (q \to (p \to r))$ (= **C**).
4. $(p \to (p \to q)) \to (p \to q)$ (= **W**).
5. $(p \to q) \to (r \to (p \to q))$ (= $\mathbf{K_1}$).
6. $(p \to p) \to ((((p \to q) \to q) \to p) \to r) \to (((((q \to p) \to p) \to q) \to r) \to r)$ (= $\mathbf{X}_4$).

   $6 * 1$ -7,

7. $((((p \to q) \to q) \to p) \to r) \to (((((q \to p) \to p) \to q) \to r) \to r)$ (= $\mathbf{X}_3$).

   $5\ q/p, r/q \to p * 1$ - 8,

8. $(q \to p) \to (p \to p)$.

   $3\ p/q \to p, q/p, r/p * 8$ - 9,

9. $p \to ((q \to p) \to p)$.

   $2\ q/p, r/(q \to p) \to p, p/(p \to q) \to q * 9$ - 10,

10. $(((p \to q) \to q) \to p) \to \mathbf{D}$.

    $3\ p/q \to r, q/p \to q, r/p \to r * 2$ - 11,

11. $(p \to q) \to ((q \to r) \to (p \to r))$ (= $\mathbf{B}'$).

    $11\ q/(q \to p) \to p, r/q * 9$ - 12,

12. $(((q \to p) \to p) \to q) \to (p \to q)$.

    $3\ p/q \to p, r/p * 1\ p/q \to p$ - 13,

13. $q \to ((q \to p) \to p)$.

    $2\ r/(q \to p) \to p, p/(p \to q) \to q) * 13$ - 14,

14. $(((p \to q) \to q) \to q) \to (((p \to q) \to q) \to ((q \to p) \to p))$.

    $2\ q/((p \to q) \to q) \to q, r/\mathbf{D}, p/p \to q * 14$ -15,

15. $((p \to q) \to (((p \to q) \to q) \to q)) \to ((p \to q) \to \mathbf{D})$.

    $15 * 13\ q/p \to q, p/q$ - 16,

16. $(p \to q) \to \mathbf{D}$.

    $2\ q/p \to q, \mathrm{r}/\mathbf{D}, p/((q \to p) \to p) \to q * 16 - 12$ - 17,

17. $(((q \to p) \to p) \to q) \to \mathbf{D}$.

7 $r/\mathbf{D} * 10$ - 17 - 18,

18. $((p \to q) \to q) \to ((q \to p) \to p)$ (= **D**).

18 $q/p \to q * 4$ - 19,

19. $((p \to q) \to p) \to p$ (= **P**) [2].

Theorem 4 is proved.

The logics **IBCWX$_4$** and **IBCK$_1$X$_4$** are of special interest. It is not hard to prove

Proposition 9. IBCWX$_4$ is **RM$_\to$**.

Further, we state the following fact.

Proposition 10. IBCK$_1$X$_4$ is **Ł** $_{\omega\to}$ (see [17, pp. 158-159] for details).

Obviously, **IBCK$_1$X$_4$** = **IBCK$_1$X$_3$**. It is routine to check that **X$_3$** is valid in the matrix for **Ł**$_\omega$. Hence by the of completeness theorem of the propositional calculus **Ł**$_\omega$ [4], **X$_3$** is provable in **Ł**$_\omega$. Since the implication $\to$ is separable in **Ł**$_\omega$ (see also Woźniakowska [48][3]), **X$_3$** is also proved in **Ł**$_{\omega\to,}$ i.e. **B', K, D** |- **X$_3$**. On the other hand, we have to show that **I, B, C, K$_1$, X$_3$** |- **K, D, L.**

(1) For **K,** the proof follows from Proposition 2.
(2) For **D**, the proof follows from Proposition 8 (see formula 18).
(3) **I, B, C, K$_1$, X$_3$** |- **L.**

1. **I**.
2. **B**.
3. **C**.
4. **K$_1$**.
5. **X$_3$**.
6. **D** (Proposition 8, formula 18).
7. $(((q \to p) \to p) \to q) \to (p \to q)$ (Proposition 8, formula 12).

   4 $p/((p \to q) \to q) \to p, q/q \to p, r/(p \to q) \to (q \to p) * 7\ q/p, p/q$ - 8,

8. $((p \to q) \to (q \to p)) \to ((((p \to q) \to q) \to p) \to (q \to p))$.

   3 $p/(p \to q) \to (q \to p), q/(((p \to q) \to q) \to p), r/q \to p * 8$ - 9,

9. $(((p \to q) \to q) \to p) \to \mathbf{L}$.

   2 $q/((q \to p) \to (p \to q)) \to (p \to q), r/\mathbf{L}, p/((q \to p) \to p) \to q * 6\ p/q \to p, q/p \to q$ -

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[2]This has been proved in collaboration with V.M.Popov. Proposition 8 (in the form **I, B, C, W, K**, **X$_3$** |– **P**) was first proved with the help of a computer program [35].

[3] In [48], **Ł**$_{\omega\to}$ is axiomatized by **K, D** and

$((p \to q) \to (p \to r)) \to ((q \to p) \to (q \to r))$.

$9\ p/q,\ q/p$ - 10,

10. $(((q \to p) \to p) \to q) \to \mathbf{L}$.

$5\ r/\mathbf{L} * 9$ -10 - 11,

11. $((p \to q) \to (q \to p)) \to (q \to p)$ (=**L**).

We have thus proved Proposition 10, and this proof was semantical. In fact, we have proved that the substitution of **D** for $\mathbf{X_3}$ in commutative **BCK**, i.e., in **BCKD**, provides an axiomatization for $\mathbf{Ł}_{\omega\to}$. Moreover, we have a purely syntactical proof of the fact that

$$\mathbf{B'KDL} \equiv \mathbf{BCKX_3} \equiv \mathbf{Ł}_{\omega\to}$$

(see [19]).

Next , we consider the lattice $L(\mathbf{TV}_\to)$ with the axiom $\mathbf{X_4}$:

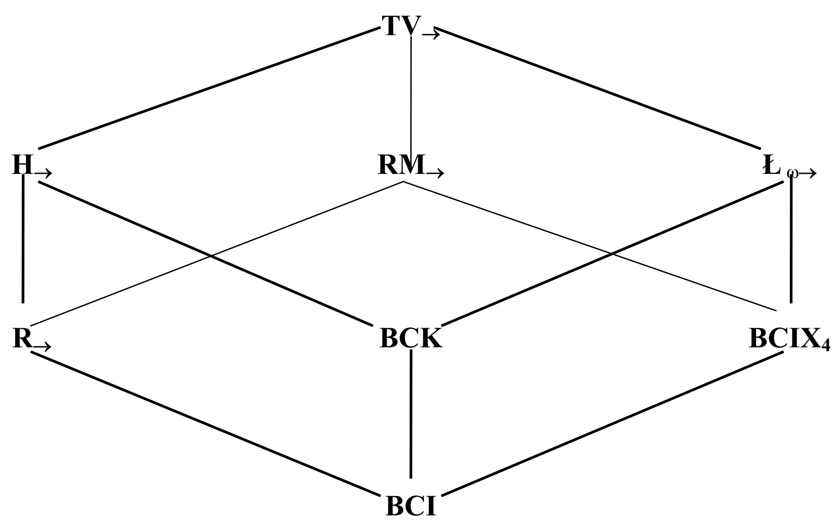

Fig. 3.

Note that $\mathbf{TV}_\to$ is the unique proper extension of $\mathbf{RM}_\to$ (see [2]). This fact immediately implies an answer to question (2) of Slaney and Bunder concerning $\mathbf{BCIWX_i}$, namely, that $\mathbf{RM}_\to$ is a stronger system. In view of this fact, such a construction is called *maximal* for implicational logics [17a]. In essence we have eight fundamental implicational logical systems.

We also have a partial answer to question (1) of Slaney and Bunder. Let α and β be arbitrary wffs. We call α *variable-like* if any propositional variable occurs in α at most once. If $\alpha = (p_1/\beta_1, p_2/\beta_2, ..., p_k/\beta_k)$, each β_i is variable-like, and for $i \neq j$, β_i and β_j have no propositional variables in common, then β is called a *restricted substitution instance* (r.s.i.) of α. Thus, $\mathbf{K_1}$ is an r.s.i. of **K**. B.Pahi established [26, Corollary 1] that if P is an extension of $\mathbf{R}_\to$ and α^* is an r.s.i. of an implicational wff α, then $P + \alpha$ and

$P + \alpha^*$ define equivalent systems. In our case, it means that any r.s.i. of formulas $\mathbf{X_1}$ - $\mathbf{X_4}$ can be chosen as $\mathbf{X}$. Thus there are infinitely many different systems $\mathbf{BCIX_i}$. For example, Let $\mathbf{X_5}$ be r.s.i. of $\mathbf{X_2}$**:** $p/p{\to}q$, q/r. Using theorem 1, we can show that **I, B, C, W, K**, $\mathbf{X_5}$ is $\mathbf{TV_{\to}}$. So $\mathbf{R_{\to} + X_2 = R_{\to} + X_5}$, but $\mathbf{BCIX_2 \neq BCIX_5}$.

Our logical construction presents natural ways for extending logics. Thus, $\mathbf{Ł_{\omega\to}}$ can be extended to $\mathbf{TV_{\to}}$ by the addition of axiom **W**. A.Rose showed in [32] that the implicational fragments $\mathbf{Ł_{n\to}}$ of Łukasiewicz's n-valued logics $\mathbf{Ł_n}$ [23] are axiomatized relative to $\mathbf{Ł_{\omega\to}}$ by an axiom

A5. $(p^{k-1} \to q) \to p) \to p$,

where k is a natural number greater than 1

For k=2, **A5** coincides with the Peirce's law **P**; for k=3, **A5** is

$\mathbf{P^3}$**.** $((p \to (p \to q)) \to p) \to p$,

and so on.

Consider the formula

A5′. $(p \to^{k} q) \to (p \to^{k-1} q)$[4].

Axiom **A5′** coincides with **W** for k=2, with

$\mathbf{W^3}$. $(p \to (p \to (p \to q))) \to (p \to (p \to q))$

for k=3, and so on.

Proposition 11. $\mathbf{Ł_{\omega\to}}$**+ A5′** is an axiomatization of $\mathbf{Ł_{n\to}}$, for every n ($2 \leq n < \omega$).

We need to show that

I. $\mathbf{Ł_{\omega\to}}$ **+ A5 |- A5′**.

II. $\mathbf{Ł_{\omega\to}}$ **+ A5′ |- A5**.

Both facts follows easily from axiom **D**.

5. The lattice of implicational logics $L(\mathbf{TV_{\to}})$: $\mathbf{E_{\to}}$, $\mathbf{S4_{\to}}$, $\mathbf{S5_{\to}}$.

We formulate the following problem. Is it possible to construct a lattice of implicational logics with $\mathbf{TV_{\to}}$ as its unit but which contain such elements as Ackermann's rigorous implication $\mathbf{E_{\to}}$, and Lewis's implications $\mathbf{S4_{\to}}$ and $\mathbf{S5_{\to}}$. In 1956, C.Meredith proved (see [20]) that $\mathbf{S5_{\to}}$ is axiomatized by **I, B′,** $\mathbf{K_1}$, $\mathbf{P_1}$, where $\mathbf{P_1}$ is

$(((p \to q) \to r) \to (p \to q)) \to (p \to q)$[5],

an r.s.i. of the Peirce's law **P.**

[4] See axiom **A4** in [43] which is introduced for axiomatization $\mathbf{Ł_n}$

[5] The axiomatization of implicational fragments of Lewis's modal systems is discussed in [12].

Implicational fragments of Lewis's modal systems (strict implication) are known not to admit the permutation law **C**. Instead they contain the formula $\mathbf{C_1}$, which is equal to the following r.s.i. of **C**:

$$(p \to ((q \to r) \to s)) \to ((q \to r) \to (p \to s)).$$

Let us consider a formula **A** $((p \to p) \to q) \to q$. Note that **AB'W** is Ackerman's implication $\mathbf{E}_\to$ [1, p. 77]. Wajsberg showed [46, p. 179] that **A,B' |- $\mathbf{C_1}$**. Note that **A |- I** [1, p. 77].

It follows from Mendez [24] that $\mathbf{AB'WK_1}$ provides an independent axiomatization of $\mathbf{S4}_\to$, an implicational fragment of **S4**. As usual $\mathbf{S5}_\to$ is considered as $\mathbf{S4}_\to$ + $\mathbf{P_1}$. Let us show that $\mathbf{S4}_\to$ + $\mathbf{X_2}$ is $\mathbf{S5}_\to$. For begininig we use Ulrich's characteristic matrix for $\mathbf{S5}_\to$ [44]. That matrix has as its values the set N of natural numbers: 1, 2, 3 …… together with 0. The sole designated value is 1. The implication $x \to y$ is defined for x and y in N, and $x \to y = 1$ if x is a multiple of y and $x \to y = 0$ otherwise. The formula $\mathbf{X_2}$ is valid in this matrix. It remains to prove

Proposition 12. **W, $\mathbf{K_1}$, $\mathbf{X_2}$ |- $\mathbf{P_1}$**.

1. **W**.
2. $\mathbf{K_1}$.
3. $\mathbf{X_2}$.

3 $p/p \to q,\ q/(p \to q) \to r * 2\ r/((p \to q) \to r) \to ((p \to q) \to r)$ -

1 $p/p \to q,\ q/(p \to q) \to r$ - 4,

4. $(((p \to q) \to r) \to (p \to q)) \to (p \to q)$ (= $\mathbf{P_1}$).

In [1a, p.46] Anderson and Belnap introduced a formula **C'**:

$$p \to ((p \to p) \to p)$$

and proved that $\mathbf{E}_\to$ **+ C' =** $\mathbf{S4}_\to$, $\mathbf{S4}_\to$ **+ C'** = $\mathbf{S5}_\to$, and $\mathbf{S5}_\to$, + **C'** = $\mathbf{TV}_\to$**.**

Theorem 5. *The set of formulas **A, B', W, K_1, X_2, C'** provides an independent axiomatization of* $\mathbf{TV}_\to$**.**

(i). The independence proof for **A** is matrix 1; for **B'** is matrix 7; for **W** is matrix 4; for $\mathbf{K_1}$ matrix 5; for $\mathbf{X_2}$ is matrix 6; for **C'** is matrix 3.

(ii). **A, B', W, $\mathbf{K_1}$, $\mathbf{X_2}$, C'** is $\mathbf{TV}_\to$ (see above).

Note that $\mathbf{X_3}$ is not valid in Ulrich's matrix (p=2, q=3, r=0).

Now we can build the lattice $L(\mathbf{TV}_\to)$ with rigorous and strict implications

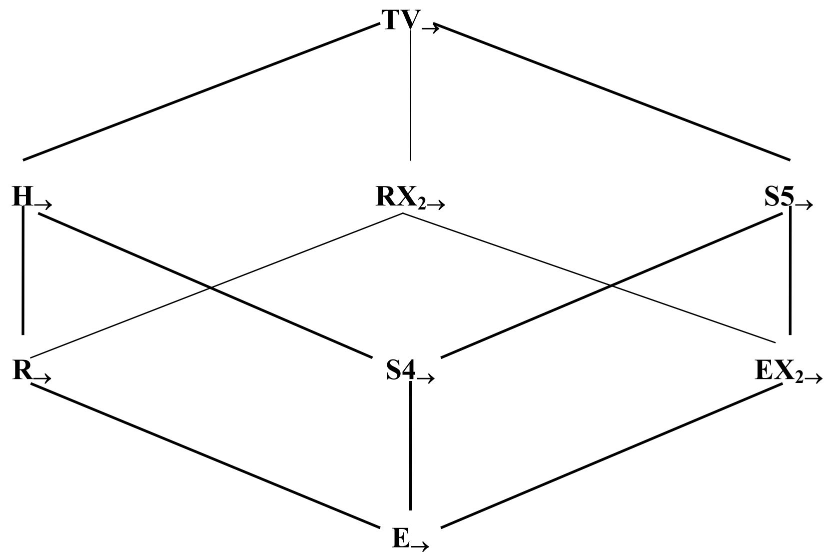

Fig. 4

6. Full propositional logics and basic principles of classification

As follows from Wajsberg's work [46, § 5], the addition of $0 \to p$ (where 0 is a constant interpreted as falsehood) to an arbitrary axiomatization of $\mathbf{TV}_{\to}$ gives the full classical propositional logic **TV**. Let us denote the formula $0 \to p$ by **N**.

Theorem 5. *The set of formulas **I, B, C, W,** $\boldsymbol{K_1}$, $\boldsymbol{X_4}$ **N** provides an an independent axiomatization of **TV**.*

(i). The independence proof for **I, B, W,** $\mathbf{K_1}$, $\mathbf{X_4}$ follows the lines of Theorem 4. The independence of **N** is proved by

Matrix 10

$\to$	0	1	2		
0	2	1	2		
1	0	2	2	verifies	falsifies
*2	0	1	2	**I,B,C,W,**$\mathbf{K_1}$**,**$\mathbf{X_4}$**,**	**N** (p=1).

(ii). **I, B, C, W,** $\mathbf{K_1}$**,** $\mathbf{X_4}$**, N** is **TV** [46] (see also [42]).

Theorem 6 is proved.

Recall that **B′**, **K, D, N** axiomatize **Ł**$_{\omega}$ [41], hence **BCKX$_2$N**, **BCKX$_3$N,** and **BCKX$_4$N** are also equal to **Ł** $_{\omega}$.

Now we can draw the lattice L(**TV**) with the logic **BCK** as zero:

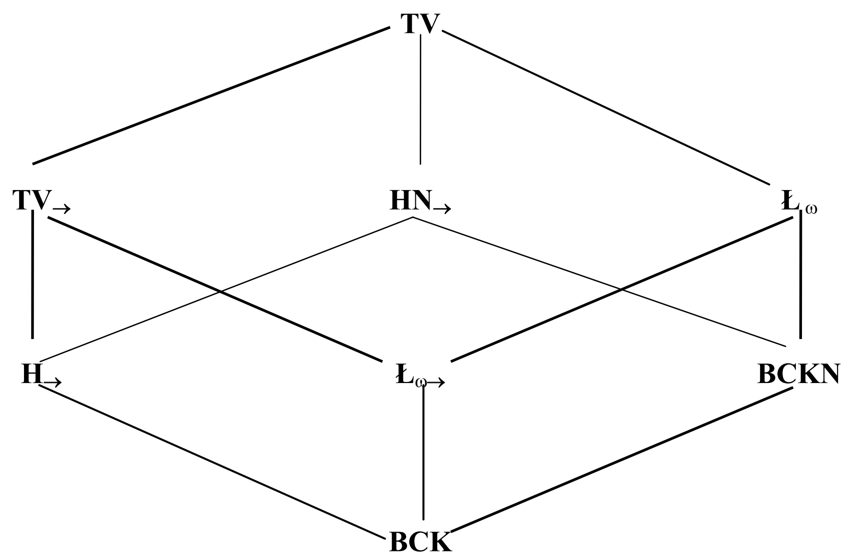

Fig. 5

Only some logics in the **TV-**construction have a lattice structure, namely, **TV** and **Ł** $_{\omega}$, but we can add a certain lattice structures to implicational logics. For example, **H**$_{\rightarrow}$ with a lattice structure is a distributive lattice [30], and together with negation $\neg p$ ($= p \rightarrow 0$) obtain the full intuitionistic propositional logic **H**.

As a result our logical construction demonstrates in an evident manner the relationships between different logics and the place these logics occupy in relation to the classical logic **TV**.

In conclusion we list the basic principles of generation of propositional logics and whole classes of them:

1. The discovery of a *new* **X$_i$** defines various sublogics generated by elements of the set {**I, B, C, W, K$_1$, X$_i$, N**}.

2. Restricted substitution generates whole classes of sublogics in the **TV$_{\rightarrow}$**-conctructions.

3. Different substitutions generate new constructions, for example, the **Ł$_n$**-constructions.

4. The combination of substitution and modus ponens gives new constructions, for example, the **S5$_{\rightarrow}$**-construction.

Thus, in some sense, we suppose that the classification of propositional calculi can be based on different constructions generating new logics.

Note. Recently a study in substructural logics becomes a new trend in logic [9a]. Usually, Gentzen's sequent calculus **LJ** is taken and different subsystems of **LJ** are constructed via varying and/or restricting of structural rules; or several restricted versions of the structural rules in the implicational fragment of Gentzen's sequent calculus **LJ** are introduced [19a]. But our approach is different in form (Hilbert calculi) as well as in content: sublogics of classical implication are considered. As result, the important implicational logics like $\mathbf{RM}_{\rightarrow}$ and $\mathbf{Ł}_{\omega\rightarrow}$ appear as well as a quite new sublogics such as $\mathbf{RX_2}$, $\mathbf{BCIX_2}$, $\mathbf{BCIX_3}$, $\mathbf{EX_2}$ and other.

ADDITION 1. In 2001 using William McCune's automated reasoning program, OTTER,[6] Vladimir Komendantsky proved that $\mathbf{B, C, W, K_1, X_3} \mid\!- \mathbf{I}$.

ADDITION 2. In 2002 Zachary Ernst suggested nine formulas **X** such that $\{\mathbf{I}, \mathbf{B, C, W, K_1, X}\}$ is an independent basis for the implicational fragment of classical logic.[7] Using OTTER program he also proved that $\mathbf{BCIX_2}$ is a subsystem of $\mathbf{BCIX_3}$ (see the problems in [35]).

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