Логические исследования
2017. Т. 23. № 2. С. 76–95
УДК 510.64

Logical Investigations
2017, Vol. 23, No. 2, pp. 76–95
DOI: 10.21146/2074-1472-2017-23-2-76-95

V.L. Vasyukov[1]

Potoses: Categorical Paraconsistent Universum for Paraconsistent Logic and Mathematic[2]

Vladimir Leonidovich Vasyukov
Department of the History and Philosophy of Science
Institute of Philosophy of Russian Academy of Sciences
12/1 Goncharnaya Str., Moscow, 109240, Russian Federation
E-mail: vasyukov4@gmail.com

It is well-known that the concept of da Costa algebra [3] reflects most of the logical properties of paraconsistent propositional calculi C_n, $1 \leq n \leq \omega$ introduced by N.C.A. da Costa. In [10] the construction of topos of functors from a small category to the category of sets was proposed which allows to yield the categorical semantics for da Costa's paraconsistent logic. Another categorical semantics for C_n would be obtained by introducing the concept of *potos* – the categorical counterpart of da Costa algebra (the name "potos" is borrowed from W.Carnielli's story of the idea of such kind of categories)

Keywords: paraconsistent logic, categorical semantics, potos, paraconsistent set theory, da Costa algebra

1. Introduction

It is well-known that the conception of da Costa algebra [3] renders the majority of the logical properties of paraconsistent propositional calculus C_n, $1 \leq n \leq \omega$ introduced by N.C.A. da Costa. In [10] the construction of the topos of functors from a small cateory to *Set* was proposed which allows to obtain the categorical semantics of da Costa's paraconsitent logics. Another categorical semantics of C_n would be introduced considering the construction of a *potos* or *da Costa topos* — a categorical equivalent of da Costa algebra.

[1]This resarch is supported by RFH grant № 16-03-00364.

[2]The paper is an improved, extended and modified English version of early published in "Logical Investigations" (vol. 17) paper "Paraconsistent Categories for Paraconsistent Logic" (in Russian).

A potos is a paraconsistent universe in which paraconsistent mathematics would be developed the way it was done in case of intuitionistic mathematics in topos. But while in case of [10] all paraconsistency instances appear only as partial constructions in the intuitionistic universe (as some local artefacts) in a potos this paraconsistency is absolutely immanent and, moreover, it underlies all the constructions, it is global and fundamental. Here the classical mathematics emerges now as an artefact in paraconsistent universe, as some local deviation from paraconsistent regularity. Thus e.g. interpreting C_n-systems one would implement non-truth-functional valuation while truth-functional valuation becomes featuring just the case of Boolean toposes which are now only the particular case of potoses.

In [11] the construction of a potos as the Cartesian closed category (with the initial object 0 and the terminal object 1) along with distinguished object Ω which is an implicit da Costa algebra was proposed, i.e. there are arrows $\boldsymbol{true} : 1 \to \Omega, \boldsymbol{false} : 1 \to \Omega, \neg : \Omega \to \Omega, \cap : \Omega \times \Omega \to \Omega, \cup : \Omega \times \Omega \to \Omega, \supset : \Omega \times \Omega \to \Omega$ which satisfy da Costa algebra conditions from $[1, p.81]$ But the shortcoming of such a definition of potos is that the arrow of negation instead of other arrows is introduced only "locally" and has no connections with any categorical constructions.

In order to overcome this shortcoming we introduce the notion of so-called *complementary closedness* of Cartesian closed category. This allows to yield the arrows of negation "globally" following the recipe of the definition of other truth-arrows.

As the consequence of the new construction introduced we need to consider a new category of paraconsistent sets $PSet$ where as the objects the ZF_1-sets of paraconsistent set theory are exploited. The system ZF_1 is correlates with Zermelo–Fraenkel set theory ZF_0 the same way the paraconsistent first-order predicate calculus with identity correlates with the classical one. And at the same time category $PSet$ turns out to be not a topos but a potos of sets.

Such type of considerations is based on J. Benabou's proposal (cf. [1]) to accept as minimal set-theoretical basis of category theory any set theory in ZF_0-language exploiting only the extensionality axiom and the comprehension scheme. For any model of such a theory its elements will be the sets while the "meta-sets" of the universe of discourse (where the model is chosen) will represent the classes. Following Benabou we can define the set theory required as any theory formulated in ZF_1 and choose any model of such a theory from the universe of discourse (in fact, extending our universe to the universe of all non-classical "meta-sets"). Then so-called Yoneda's map will assign to every set its representative class relative to our "paraconsistent" model while our sets will be exactly the sets we need for considering $PSet$.

In section 2 the minimal information concerning the theory of da Costa algebras is adduced which is required for the further considerations since the last are essentially exploited and substantially determine the potos construction itself.

In section 3 the notion of a potos is introduced, its properties and set-theoretical foundations are analyzed and the potos $PSet$ is considered.

In section 4 an algebraic interpretation of da Costa systems $C_n$in terms of da Costa algebra is yielded along with potos-theoretic interpretation of those essentially based on this algebra. The completeness of the systems is proved in respect to the interpretation given. Besides, non-truth-functional valuation of C_1 is considered and the completeness of this system is proved exploiting this valuation.

Finally in section 5 the interpretation in potos $PSet^A$ is considered and the completeness of C_1 in respect to such semantics is proved.

2. Da Costa Algebras

W.A. Carnielli and L.P. Alcantara in 1984 [3] formulated the notion of da Costa algebra reflected the most of logical properties of logic C_n. It was shown that da Costa algebra is isomorphic to a paraconsistent set algebra which would be counted as an counterpart of Stone representation theorem for Boolean algebra. However, such an analogy works only if we takes a non-classical point of view: some operations in paraconsistent set algebra are formulated not in usual set-theoretical terms.

Since our theoretic-categorical constructions are essentially based on da Costa algebra then for the further proceedings the complete definitions are adduced.

Definition 1. [3, p. 81] By a da Costa algebra we mean a structure

$A = \langle S, 0, 1, \leq, \wedge, \vee, \supset, ' \rangle$

such that for every a, b, c in S the following conditions hold:

1. $\leq$ is a quasi-order;
2. $a \wedge b \leq a, a \wedge b \leq b$;
3. if $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$;
4. $a \wedge a = a, a \vee a = a$;
5. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$;
6. $a \leq a \vee b, b \leq a \vee b$;
7. if $a \leq c$ and $b \leq c$ then $a \vee b \leq c$;
8. $a \wedge (a \supset b) \leq b$;
9. if $a \wedge c \leq b$ then $c \leq (a \supset b)$;
10. $0 \leq a, a \leq 1$;
11. $x^o \leq (x')^o$, where $x^o = (x \wedge x')'$;
12. $x \vee x' \equiv 1$, where $a \equiv b$ iff $a \leq b$ and $b \leq a$;
13. $x'' \leq x$, where x'' abbreviates $(x')'$;
14. $a^o \leq (b \supset a) \supset ((b \supset a') \supset b')$;
15. $x^o \wedge (x^o)' \equiv 0$.

If there exists $x \in S$ such that it is not true that $x \wedge x' \equiv 0$ the algebra A is said to be *a proper da Costa algebra*.

Let us note that it would be much more natural to consider congruences instead of equations in items 4 and 5 in definition 1. This is the choice made e.g. by Carlos Caleiro and Ricardo Gonçalves developed so-called Behavioral algebraization of da Costa's C-systems (cf. [2]).

Proposition 1. [3, p. 82] *If $A = \langle S, 0, 1, \leq, \wedge, \vee, \supset, ' \rangle$ is a da Costa algebra then the following properties are verified:*

($C1$) *$y \leq x$ iff $x \wedge y \equiv y$;*

($C2$) *$x \wedge 0 \equiv 0, x \vee 1 \equiv 1$;*

($C3$) *$x \vee 0 \equiv x, x \wedge 1 \equiv x$;*

($C4$) *$x \vee y \equiv y \vee x, x \wedge y \equiv y \wedge x$;*

($C5$) *if $x = y$ then $x \equiv y$;*

($C6$) *if $a \equiv b$ and $x \equiv y$ then $x \wedge a \equiv y \wedge b$;*

($C7$) *if $a \equiv b$ and $x \equiv y$ then $x \vee a \equiv y \vee b$;*

($C8$) *$y \leq x$ iff $y \vee x \equiv x$;*

$(C9)$ *if* $x \vee y \equiv 0$ *then* $x \equiv 0$ *and* $y \equiv 0$;
$(C10)$ *if* $x \leq y^o$ *and* $x \leq (y^o)'$ *then* $x \equiv 0$;
$(C11)$ $p \vee (p' \wedge p^o) \equiv 1$, $p' \vee (p' \wedge p^o) \equiv 1$;
$(C12)$ $x \wedge y \equiv 0$ *iff* $x \leq y' \wedge y^o$;
$(C13)$ $x \wedge (y \wedge y^o) \equiv 0$ *iff* $x \leq y'$;
$(C14)$ $x \wedge x' \wedge x^o \equiv 0$;
$(C15)$ *if* $x \wedge (x')^o \equiv 0$ *then* $x \equiv x''$;
$(C16)$ *if* $x' \wedge (x')^o \equiv 0$ *then* $x \equiv x''$;
$(C17)$ *if* $x \wedge y \equiv 0$ *then* $x \leq y'$;
$(C18)$ *if* $y \equiv y^o$ *then* $x \wedge y \equiv 0$ *iff* $x \leq y'$;
$(C19)$ $x \wedge y' \wedge y^o \equiv 0$ *iff* $x \leq y$.

PROOF. Obvious. □

Hereafter □ means the end of the proof.

THEOREM 1. [3, p. 83] *Every proper da Costa algebra has at least three elements.*

DEFINITION 2. [3, p. 83] A paraconsistent algebra of sets is a structure
$A = \langle S, \varnothing, I, \leq \cap, \cup, \Rightarrow, ' \rangle$
where
1. $\cap$ and $\cup$ are the set operations of intersection and union;
2. $\leq$ is a preorder;
3. $S \subseteq \wp(I)$;
4. S is closed with respect to the binary operations $\cap$, $\cup$, and the unary operation $'$;
5. $a \cap b \leq a$, $a \cap b \leq b$;
6. if $c \leq a$ and $c \leq b$ then $c \leq a \cap b$;
7. $a \leq a \cup b, b \leq a \cup b$;
8. $a \cap (a \Rightarrow b) \leq b$;
9. if $a \cap c \leq b$ then $c \leq (a \Rightarrow b)$;
10. $\varnothing \leq a, a \leq I$;
11. $x \cup x' \Leftrightarrow I$, where $a \Leftrightarrow b$ iff $a \leq b$ and $b \leq a$;
12. $x'' \leq x$;
13. $x^o \leq (y \Rightarrow x) \Rightarrow ((y \Rightarrow x') \Rightarrow y')$, where $x^o = (x \cap x')'$;
14. $x^o \cap (x^o)' \Leftrightarrow \varnothing$;

15. $x^o \leq (x')^o$;

Let $S_0 = \{x : x \in S \text{ and } x \cap x' \nLeftrightarrow \varnothing\}$. If $S_0 \neq \varnothing$, we have a *proper paraconsistent algebra of sets.* Every paraconsistent algebra of sets is a proper da Costa algebra while every Boolean algebra of sets is a non-proper paraconsistent algebra of sets.

If one consider a natural notion of congruence on da Costa algebra and then to define the notion of homomorphic image of a da Costa algebra (= isomorphism (projection on quotient by a congruence)) then the following result obviously will take place:

Theorem 2. *Every proper da Costa algebra is isomorphic to a quotient to a proper paraconsistent algebra of sets.*

In [3, p. 84], in fact, more weaker notion of isomorphism was considered which is not symmetric indeed. Given an equivalence relation $\sim$ two da Costa algebras A and B are said to be $\sim$-isomorphic if there exists a function f from A onto B preserving the operations and being $\sim -$*injective* that is, if $x \nsim y$ then $f(x) \neq f(y)$. And the following result take place

Theorem 3. [3, p. 84] *Every proper da Costa algebra is* $\equiv -$*isomorphic to a proper paraconsistent algebra of sets.*

In [9, p. 273] the following theorem was proved:

Theorem 4. *A set of principal filters of the proper da Costa algebra is* $\equiv -$*isomorphic to a proper da Costa algebra.*

Taking into account that every principal filter is determined by the single element of a da Costa algebra then this $\equiv -$isomomorphism will be symmetric one.

3. Potoses

A potos is, in fact, a topos with some additional structure. In essence, we would equally well use the name “paraconsistent topos” or “da Costa topos”. The name “potos” was borrowed from W.Carnielli’s story of the idea of such kind of categories originated from some Brasilian mathematician.

Definition 3. A **potos** C is a Cartesian closed category which is also complementary closed and has a subobject classifier. That is:

(i) C has finite products $\langle -,-\rangle$,$[-,-]$ and C is distributive relative to those, i.e. $\langle [a,b],[a,c]\rangle \cong [a,\langle b,c\rangle]$ for any objects a,b,c in C;

(ii) C allows an exponentiation;

(iii) C has a terminal object 1 and an initial object 0;

(iv) $a\to b$ is an arrow in C iff $a\Rightarrow b\cong 1$, for any two objects a,b in C where $a\Rightarrow b$ is an exponential;

(v) for any object a of C there is an object a' with the respective operations (functions) in C :

$()' : obj(C)\to obj(C)$ such that $a\mapsto a'$,

$dn : obj(C)\to Hom(C)$ such that $a\mapsto dn(a) : a''\to a$,

$hdn : Hom(C)\to Hom(C)$ such that $d\to a\mapsto hdn(a,d) : d\to a'$,

$()^o : obj(C)\to obj(C)$ such that $a\mapsto a^o=\langle a,a'\rangle'$,

$cert : obj(C)\to Hom(C)$ such that $a\mapsto cert(a) : a^o\to (a')^o$,

$hcert : Hom(C)\to Hom(C)$ such that $d\to a^o\mapsto hcert(a,d) : d\to (a')^o$,

where $dn(a)$ and $cert(a)$ are monic and we have a fixed choice of coproducts $[-,-]$ and of products $\langle -,-\rangle$ for each pair of objects in the respective operations in C;

(vi) there is an operation $triv : obj(C)\to Hom(C)$ in C such that $a\mapsto triv(a) : a^o\to (b\Rightarrow a)\Rightarrow ((b\Rightarrow a')\Rightarrow b')$ and $triv(a)$ is monic;

(vii) $1\cong [a,a'], 0\cong\langle a^o,a^{o\prime}\rangle$(with, possibly, binary coproducts and products);

$(viii)$ a subobject classifier for C is a C-object Ω together with an arrow $true : 1\to\Omega$ that satisfies the following axiom: for each monic $f : a\rightarrowtail d$ there is one and only one arrow $\chi_f : d\to\Omega$ such that

$$\begin{array}{ccc} a & \overset{f}{\rightarrowtail} & d \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \chi_f} \\ 1 & \underset{true}{\longrightarrow} & \Omega \end{array}$$

is a pullback square.

Here complementary closedness of C is given by $(v)-(vii)$. To put this another way, complementary closedness is given by:

(v) *for any object* a *of* C *there is an object* a', *such that*:

for any arrow $f : d \to a$ *we have monos* $f' : d \to a', dn(f) : a'' \to a$ *and* $cert(f) : a^o \to (a')^o$ *where* $a^o = \langle a, a' \rangle'$,

(vi) *for any two objects* a, b *in* C *there is a mono* $triv(a, b) : a^o \to (b \Rightarrow a) \Rightarrow ((b \Rightarrow a') \Rightarrow b')$;

(vii) $1 \cong [a, a'], 0 \cong \langle a^o, a^{o\prime} \rangle$(*with, possibly, binary coproducts and products*).

Proposition 2. *In a potos* C *the set* $Sub(d)$ *of subobjects of* d *(= set of equivalence classes of monos with codomain* d*) is a da Costa algebra.*

Proof. Since any potos C is a Cartesian closed category then for any object d in C the collection $Sub(d)$ of all C-arrows that are monic with d as codomain will be preordered bounded distributive lattice. This gives us that the conditions 1-7 and 10 of the definition of da Costa algebra are fulfiled in $Sub(d)$. The conditions 8-9 are the consequences of the exponentiation diagram. The conditions (v)-(vii) of the definition of potos provides us the conditions 11–15 are to be held. □

It is easy to see that in potos we have $Sub(d) \cong Hom(d, \Omega)$ and thus $Hom(d, \Omega)$ will be a da Costa algebra. But in this case the problem arises concerning the category Set of sets. The matter of fact is that in Set we have $Sub(D) \cong \wp(D)$ where $\wp(D) = \{x : x$ is a subset of the set $D\}$ Since $\wp(D)$ is a Boolean algebra of subsets and not the paraconsistent algebra of sets as we can expect from the theorem 5, then we come to the conclusion that Set cannot be a potos. But according the definition of a paraconsistent algebra of sets there are some sets which form such an algebra. So either such sets generates the subcategory $PSet$ of Set or Set is, in a sense, a subcategory of $PSet$.

It is known (cf. [5]) that there is a system ZF_1 of paraconsistent set theory that related to Church's version of Zermelo–Fraenkel set theory ZF_0 with a universal set as a da Costa paraconsistent first-order logic $C_1^=$ is related to the clasical first-order predicate calculus $C_0^=$. In essence, "ZF_1 should be 'partially' included in ZF_0, though the latter is is also to be contained, in a certain sense, in the former" [5, p. 170]. The basic set-theoretic concepts of ZF_1 are analogous to those of ZF_0, although the concepts involving negation give rise to two notions: one involving the weak negation ($\neg$) and the other the strong negation ($\neg*$). As a result we have, for instance, two empty sets: $\varnothing = \{x : x \neq x\}$ and $\varnothing^* = \{x : \neg * (x = x)\}$.

The collection of all sets, plus $\varnothing, V, \cap, \cup, ^C$ (where $V = \{x : x = x\}$ and $x^C = \{y : y \notin x\}$) form in ZF_0 a complete Boolean algebra. For ZF_1 we obtain the following result:

Proposition 3. *In ZF_1 the collection of all sets, plus $\varnothing^*, V, \cap, \cup, ^{C^*}$ form a paraconsistent algebra of sets.*

Proof. By immediate checking (putting $x \Rightarrow y = \{z : z \in x \to z \in y\}$). □

Each axiom scheme of ZF_0 generates two corresponding axiom schemes of ZF_1, one with the strong negation and another with the weak one. Thus, we can say that ZF_1 includes ZF_0 and hence, *Set* is actually, in a sense, a subcategory of *PSet*. But what does it means to be a category of sets other than *Set*?

Shepherdson in 1952 [9] introduced a notion of an "inner model" of a logic L. He means a model whose universe is a subset of the universe of L, and in which the "true" statements are those statements of the model which are provable in L. Shepherdson assumes that the universe of L contains classes, sets, and possibly additional objects. The models have classes and sets and a membership relation $\in_m$. For every non-empty class A of L, there is a member $y \in A$ such that $z \in_m y$ for every member z of A.

Later J. Benabou in [1, p. 18] trying to find the minimal set theoretical foundations for category theory defines *a set theory* as any theory T written in the language of Zermelo–Fraenkel set theory ZF and satisfying at least the extensional axiom E and the comprehension scheme CS.

Let us for any model M of such a theory the elements of M will be called sets and denoted by $S, T, ...$ and the formal membership and equality of sets will be denoted by $S \in^* T$ and $S \doteq T$. Then the "metasets" of the universe of discourse $\mathbf{U}$ where the model M is taken will be called classes and denoted by **S**,**C**,... while the membership and equality in **U** is denoted by the usual notations $\in$ and $=$. A subclass **S** of M will be *representable* if there is a set S, called a *representative* of **S**, such that for all $T \in M$ we have $T \in \mathbf{S}$ iff $T \in^{*} S$. The *Yoneda map* assign to each set S the representative class $\hat{S} = \{T \in M : T \in^{*} S\}$

The extensionality axiom, for example, read thus:

(E) For all sets S and T, $\hat{S} = \hat{T}$ iff $S \doteq T$.

Unfortunately, it is known that such naive set theory is inconsistent and the source of it is the exploitation of classical logic underlying such theory. To overcome this troublesome case we can e.g. use in the role of T Grishin's LST theory (cf. [8]) which is the set theory with the unlimited comprehension scheme based on the modified linear Girard's logic and which is free of the mentioned shortcomings. But in our case it does not matter since we have to consider paraconsistent set theory and we are concerned just with the universe of discourse **U** which is hypothetically contains any kind of sets.

Following Benabou's course we can now define a set theory as any theory T written in the language of ZF_1 and choose any model M of such a theory from the universe of discource **U** (in fact, extending it to the universe of all non-classical "meta-sets"). Then Yoneda map will assign to each set S the representative class $\hat{S} = \{T \in M : T \in^{*} S\}$ relative to our "paraconsistent" model M and our sets will be exactly the sets we need for considering the category $PSet$.

PROPOSITION 4. *$PSet$ is a potos.*

PROOF. According to proposition 10 for any set I we always have [3, p. 84] a paraconsistent algebra of sets $\langle S, \varnothing, I, \leq \cap, \cup, \Rightarrow, ' \rangle$ where $\cap$ and $\cup$ are the set operations of intersection and union, $\leq$ is a preorder, $S \subseteq \wp(I)$, S is closed with respect to the $\cap$, $\cup$, and the unary operation $'$. If we consider inclusion functions as arrows then we can define $x \leq y$ iff $x \hookrightarrow y \cup \{b\}$, where $b \in S$. We define $x' = x^c$ if $x \notin S_0$ and $x' = x^c \cup \tau$ if $x \in S_0$, taking $S_0 = \{x \in S$: there exists $\tau = \{a, b\}$ such that $x \cap \tau \neq \varnothing, x^c \cap \tau \neq \varnothing$ and $\neg(x \subset \{a, b\})\} \neq \varnothing$, x^c being the set-theoretical complement of x. Finally, we define $x \Rightarrow y$ is $x' \cup y$.

It is easy to see that in our algebra $x^o = I$ if $x \notin S_0$ and $x^o = I - \{b\}$ if $x \in S_0$, for $b \in x \cap \tau$. Also $\leq$ is a proper preorder for if $b \in x$ then we have $x \cup \{b\} \leq x$ and $x \leq x \cup \{b\}$ but $x \cup \{b\} \neq x$. Hence, defining $x \Leftrightarrow y$ iff $x \leq y$ and $y \leq x$ we get an equivalence relation other than equality. Moreover, if $x \subseteq y$ (and thus there is an inclusion arrow $x \hookrightarrow y$) then $x \leq y$ and $x = y$ imply $x \Leftrightarrow y$. Our equivalence relation is $\Leftrightarrow$ is not compatible with $'$, since if we take x such that $\neg(\tau \subset x)$ then $x \cup \tau \Leftrightarrow x \cup \{a\}$ where $\tau = \{a, b\}$. But $(x \cup \tau)' = (x \cup \tau)^c = x^c - \tau$ and $(x \cup \{a\})' = (x \cup \{a\})^c \cup \tau = x^c = \tau$. Thus, $x^c - \tau \not\Leftrightarrow x^c \cup \tau$.

So, we can conclude that in $PSet$ we have $Sub(d) \cong \wp(d)$ and $Sub(d)$ will be a paraconsistent set algebra and so do $\wp(d)$. But in this case we cannot take 2 as the classifying object exploiting the fact that$\wp(d) \cong 2^d$ because this gives rise to the Boolean algebra of characteristic arrows as in Set. Actually, if we will try to define

$$\chi_A(x) = \{1, \text{ if } x \in A0, \text{if } x \notin A$$

then we need to take into account that in $PSet$ we have two negations and hence the right definition will be

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 2, \text{if } x \notin A \\ 0, \text{ if } \neg * (x \in A) \end{cases}$$

This means that in the role of classifying object in $PSet$ we should take not the two-element Boolean algebra but the three-element da Costa algebra (according theorem 3 every proper da Costa algebra has at least three elements). An example of such an algebra would be found in [1, p. 83] where the operations are defined by the following tables:

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\supset$	0	1	2
0	1	1	1
1	0	1	0
2	0	1	1

$0' = 1,\ 1' = 0,\ 2' = 1;\ 0 \leq 2 \leq 1.$

So, we have $\wp(d) \cong 3^d$ and the set $3 = \{\varnothing^*, \varnothing, \{\varnothing^*\}\}$ together with the function $true : 1 \to 3$ such that $true(\varnothing^*) = 1$ (where $1 := \{\varnothing^*\}$)

playing the role of the subobject classifier in $PSet$. Also here we have arrows, $false : 1 \to 3$ (such that $false(\varnothing^*) = \varnothing$) and $false^* : 1 \to 3$ (such that $false^*(\varnothing^*) = \varnothing^*$). □

We define now truth-arrows in potos in general case. Let us **C** will be a potos with the subobject classifier $true : 1 \to \Omega$. Then the negation $\neg : \Omega \to \Omega$ will be the unique arrow for which the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{false} & \Omega \\ \downarrow & & \downarrow \neg \\ 1 & \xrightarrow[true]{} & \Omega \end{array}$$

will be the pullback in **C**. Thus, $\neg = \chi_{false}$. The negation $\neg^* : \Omega \to \Omega$ will be the unique arrow for which the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{false*} & \Omega \\ \downarrow & & \downarrow \neg* \\ 1 & \xrightarrow[true]{} & \Omega \end{array}$$

will be the pullback in **C**. In this case we have $\neg^* = \chi_{false^*}$

1. Since potos is a Cartesian closed category then other truth-arrows will be defined standardly:

$\cap : \Omega \times \Omega \to \Omega$ is a character of the product of arrows $\langle true, true \rangle : 1 \to \Omega \times \Omega$ in a potos **C**;

$\cup : \Omega \times \Omega \to \Omega$ is by definition a character of the image of **C**$-$**arrow** $[\langle true_\Omega, 1_\Omega \rangle, \langle 1_\Omega, true_\Omega \rangle] : \Omega + \Omega \to \Omega \times \Omega$;

$\Rightarrow : \Omega \times \Omega \to \Omega$ is a character of the monic $c : \circledast \rightarrowtail \Omega \times \Omega$, which is an equalizer of the pair

$$\Omega \times \Omega \ \underset{pr_1}{\overset{\cap}{\rightrightarrows}} \ \Omega$$

where pr_1 is a projection on the first member of the product $\Omega \times \Omega$.

4. An Interpretation of Paraconsistent Logic in a Potos

Let us give an interpretation in terms of potoses of the following list of axioms and rule of inference [4, p. 3790]:

$A1.\ \alpha \supset (\beta \supset \alpha)$
$A2.\ (\alpha \supset \beta) \supset ((\alpha \supset (\beta \supset \gamma)) \supset (\alpha \supset \gamma))$
$A3.\ \alpha \wedge \beta \supset \alpha$
$A4.\ \alpha \wedge \beta \supset \beta$
$A5.\ \alpha \supset (\beta \supset \alpha \wedge \beta)$
$A6.\ \alpha \supset \alpha \vee \beta$
$A7.\ \beta \supset \alpha \vee \beta$
$A8.\ (\alpha \supset \gamma) \supset ((\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma))$
$A9.\ \alpha \vee \neg\alpha$
$A10.\ \neg\neg\alpha \supset \alpha$
$A11.\ \beta^o \supset ((\alpha \supset \beta) \supset ((\alpha \supset \neg\beta) \supset \neg\alpha))$
$A12.\ \alpha^o \wedge \beta^o \supset (\alpha \wedge \beta)^o$
$A13.\ \alpha^o \wedge \beta^o \supset (\alpha \vee \beta)^o$
$A14.\ \alpha^o \wedge \beta^o \supset (\alpha \supset \beta)^o$
$A15.\ \alpha^o \supset (\neg\alpha)^o$
$R1.\ \dfrac{\alpha \quad \alpha \supset \beta}{\beta}$

Here α^o is an abbreviation for $\neg(\alpha \wedge \neg\alpha)$.This axiomatic describes, in fact, the system C_1 of da Costa paraconsistent logic.

We can define a valuation $v : \Phi_0 \to A$ of the system C_1 in da Costa algebra A assigning to an every propositional letter π_i some truth-value $V(\pi_i) \in A$. It uniquely would be extended in a following way:

(1) $v(\neg\alpha) = v(\alpha)'$;
(2) $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$;
(3) $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$;
(4) $v(\alpha \supset \beta) = v(\alpha) \supset v(\beta)$.

to the function $v : \Phi \to A$. The sentence α such that $v(\alpha) = 1$ for every A-valuation v is called A-valid and this is denoted as $A \models \alpha$.

THEOREM 5. *For any da Costa algebra A, $A \models \alpha$ iff $\vdash_{C_1} \alpha$.*

PROOF. From left to right we check immediately C_1-validity of all C_1-axioms and detachement rule. For obtaining the proof of the claim from right to left we will use the theorem 6. Putting into the correspondence to each element x of an algebra A the principal filter $[x) = \{q : x \leq q\}$ where $q \in A$ we come to the conclusion that an algebra A^+ of principal filters will be $\equiv$ -isomorphic da Costa algebra. Let us define A^+-valuation as a function $v_c : \Phi_0 \to A^+$ by means of the formula $v_c(\pi_i) = [v(\pi_i))$. The rest is obvious.

There is one more way to prove this theorem if we will use the non-truth-functional valuation of C_1. Following [6] we can introduce a valuation $V' : \Phi_0 \to A$ where Φ_0 is a set of propositional letters and extend this to the set Φ of all formulas in the following way:

(5) $V'(\alpha) = 0 \Rightarrow V'(\neg\alpha) = 1$;

(6) $V'(\neg\neg\alpha) = 1 \Rightarrow V'(\alpha) = 1$;

(7) $V'(\beta^\circ) = V'(\alpha \supset \beta) = V'(\alpha \supset \neg\beta) = 1 \Rightarrow V'(\alpha) = 0$;

(8) $V'(\alpha \supset \beta) = 1 \Leftrightarrow V'(\alpha) = 0$ or $V'(\beta) = 1$;

(9) $V'(\alpha \wedge \beta) = 1 \Leftrightarrow V'(\alpha) = V'(\beta) = 1$;

(10) $V'(\alpha \vee \beta) = 1 \Leftrightarrow V'(\alpha) = 1$ or $V'(\beta) = 1$;

(11) $V'(\alpha^\circ) = V'(\beta^\circ) = 1 \Rightarrow V'((\alpha \supset \beta)^\circ) = V'((\alpha \wedge \beta)^\circ) = V'((\alpha \vee \beta)^\circ) = 1$.

According to [6, p. 623] $A \models \alpha$ iff $\vdash_{C_1} \alpha$ i.e. α is valid for every valuation V'. □

Let us define now an interpretation of the system considered in an arbitrary potos C. The truth-value in potos we will call an arrow of the type $1 \to \Omega$ and the collection of all such C-arrows will be the set $C(1, \Omega)$.

C-valuation will be a function $V : \Phi_0 \to C(1, \Omega)$ assigning to an every propositional variable π_i some truth-value $V(\pi_i) : 1 \to \Omega$. This function might be extended to the set Φ of all formulas in the following way:

(12) $V(\alpha) = false \Rightarrow V(\neg\alpha) = true$:

(13) $V(\neg\neg\alpha) = true \Rightarrow V(\alpha) = true$;

(14) $V(\beta^\circ) = V(\alpha \supset \beta) = V(\alpha \supset \neg\beta) = true \Rightarrow V'(\beta^\circ) = V'(\alpha \supset \beta) = V'(\alpha \supset \neg\beta) = 1$;

(15) $V(\alpha \supset \beta) = true \Leftrightarrow V(\alpha) = false$ or $V(\beta) = true$;

(16) $V(\alpha \wedge \beta) = true \Leftrightarrow V(\alpha) = V(\beta) = true$;

(17) $V(\alpha \vee \beta) = true \Leftrightarrow V(\alpha) = 1$ or $V(\beta) = true$;
(18) $V(\alpha^\circ) = V(\beta^\circ) = true \Rightarrow V'(\alpha^\circ) = V'(\beta^\circ) = 1$.

Thus, we extend the valuation V in such a way that to each sentence α corresponds some C-arrow $V(\alpha){:}1 \to \Omega$. C-validity of α (which is denoted $C \models \alpha$) means that $V(\alpha) = true : 1 \to \Omega$ for all V.

Since a potos is a particular kind of topos, we have $Sub(d) = C(1, \Omega^d)$ then $Sub(d) = C(d, \Omega)$ ($Sub(d)$ and not its quotient is $\equiv$-isomorphic to $C(d, \Omega)$, using the definition of $\equiv$-isomorphism given above), i.e. taking into correspondence to some subobject f its character χ_f we transfering the structure of da Costa algebra from $Sub(d)$ on $C(d, \Omega)$. The connection between potos semantics and considered theory as in case of Heyting algebra (cf. [7]) consists in that for any potos

$$C \models \alpha \text{ iff } C(1, \Omega) \models \alpha \text{ iff } Sub(1) \models \alpha$$

Hence, the validity in any potos C is equal to the validity in da Costa algebras $C(1, \Omega)$ and $Sub(1)$. This implies the following theorem:

THEOREM 6. *If* $\vdash_{C_1} \alpha$ *then for any potos* C *we have* C $\models \alpha$.

PROOF. Let α be some C_1-theorem. Then α is valid in da Costa algebra by theorem 12. In particular, $C(1, \Omega) \models \alpha$ from which $C \models \alpha$ according to the previous claim.

We would define the way V relates to V' from above while putting $V(\pi_i) = true$ if $V'(\pi_i) = 1$, and $V(\pi_i) = false$ otherwise. Then we extend this to the set Φ of all formulas in the following way:

(19) $V(\alpha) = false \Leftrightarrow V'(\neg\alpha) = 1$:
(20) $V(\neg\neg\alpha) = true \Leftrightarrow V'(\neg\neg\alpha) = 1$;
(21) $V(\beta^\circ) = V(\alpha \supset \beta) = V(\alpha \supset \neg\beta) = true \Leftrightarrow V'(\beta^\circ) = V'(\alpha \supset \beta) = V'(\alpha \supset \neg\beta) = 0$;
(22) $V(\alpha \supset \beta) = true \Leftrightarrow V'(\alpha \supset \beta) = 1$;
(23) $V(\alpha \wedge \beta) = true \Leftrightarrow V'(\alpha \wedge \beta) = 1$;
(24) $V(\alpha \vee \beta) = true \Leftrightarrow V'(\alpha \vee \beta)$;
(25) $V(\alpha^\circ) = V(\beta^\circ) = true \Leftrightarrow V'((\alpha \supset \beta)^\circ) = V'((\alpha \wedge \beta)^\circ) = V'((\alpha \vee \beta)^\circ) = 1$.

It is easy to prove that $V(\alpha) = true$ iff $V'(\alpha) = 1$ that allows us to obtain the proof of

LEMMA 1. $V(\alpha) = true$ iff $V'(\alpha) = 1$.

PROOF. In case $\alpha = \pi_i$ lemma is true by the definition.

Let $\alpha = \neg\beta$ and $V(\beta) = false$. Then $V'(\neg\beta) = 1$ and $V'(\alpha) = 1$ and the other way round.

In case of $\alpha = \neg\neg\beta$ and $V(\neg\neg\beta) = true$ we have $V'(\neg\neg\beta) = 1$ and $V'(\alpha) = 1$.

For $\alpha = \beta^\circ$ and $V(\beta^\circ) = V(\alpha \supset \beta) = V(\alpha \supset \neg\beta) = true$ we have $V'(\beta^\circ) = V'(\alpha \supset \beta) = V'(\alpha \supset \neg\beta) = 0$ and thus $V'(\alpha) = 0$.

For $\alpha = \gamma \supset \beta$ we have $V(\alpha \supset \beta) = true \Leftrightarrow V'(\alpha \supset \beta) = 1$.

In case of $\alpha = \gamma \wedge \beta$ we have $V(\gamma \wedge \beta) = true \Leftrightarrow V'(\gamma \wedge \beta) = 1$.

In case of $\alpha = \gamma \vee \beta$ we have $V(\gamma \vee \beta) = true \Leftrightarrow V'(\gamma \vee \beta) = 1$.

Finally, taking $V(\alpha^\circ) = V(\beta^\circ) = true$ we obtain $V'((\alpha \supset \beta)^\circ) = V'((\alpha \wedge \beta)^\circ) = V'((\alpha \vee \beta)^\circ) = 1$.

The rest we obtain in a similar way. □

□

THEOREM 7. *For any potos* C *and propositional fromula* α *the following statement is true:*

C $\models \alpha$ *iff* $\vdash_{C_1} \alpha$.

PROOF. Suppose $\nvdash_{C_1} \alpha$ then, by the completeness result in [6], there is a valuation V' such that $V'(\alpha) \neq 1$ and, by Lemma 4.3, the associated V is such that $V(\alpha) \neq true$, and this means $C \nvDash \alpha$. □

5. An Interpretation of Paraconsistent Logic in a Potos $PSet^A$

For obtaining an interpretation of C_1 in a topos Set^A in [10] as the categorical counterpart of da Costa algebra so-called CN-categories have been implemented. But since $x \leq y \Rightarrow y' \leq x'$ is not a valid property concerning the paraconsistent negation in C_1 then we need to reformulate the definition of CN-categories.

DEFINITION 4. A CN-category C is a preorder category such that

(i) C has finite products $\langle -, - \rangle$, coproducts $[-, -]$ and C is distributive relative to those, i.e. $\langle [a, b], [a, c] \rangle \cong [a, \langle b, c \rangle]$ for any objects a, b, c in C;

(ii) C allows an exponentiation;

(iii) $a \to b$ is an arrow in C iff $a \Rightarrow b \cong 1$, for any two objects a, b in C where $a \Rightarrow b$ is an exponential;

(iv) C has a terminal object 1 and an initial object 0;

(v) for any object a of C there is an object Na such that we have arrows $NNa \to a$ and $a^o \to (Na)^o$ in C where $a^o = N\langle a, Na\rangle$ and for any arrow $d \to a$ there is an arrow $d \to Na$ in C;

(vi) for any two objects a, b in C there is an arrow $a^o \to (b \Rightarrow a) \Rightarrow ((b \Rightarrow Na) \Rightarrow Nb)$;

(vii) $1 \cong [a, Na]$ and $0 \cong \langle a^o, Na^o\rangle$.

It is easy to check that any CN-category has the following properties:a

an exponential $a \Rightarrow b$ in C will be a residual,

C is cartesian closed,

$y \to x$ is an arrow in C iff $\langle x, y\rangle \cong y$ and $[x, y] \cong x$,

$\langle\langle \mathbf{N}a, a^o\rangle, a\rangle \cong 0, [\langle \mathbf{N}a, a^o\rangle, a] \cong 1$,

every CN-category has at least three objects.

In order to build the category $PSet^A$ as a potos we will use the theorem 7. According to this theorem a set A^+ of all principal filters i.e. of sets $[p) = \{q : p \leq q\}$ is a da Costa algebra $\equiv$-isomorphic to A and this will be true for $[p)^+$ where $[p)^+$ is the relativization of A^+.

Now we consider the functor $\mathbf{\Omega} : A \to PSet$ which will represent the classifying object in potos $PSet^A$. Hereafter we will use A both as an algebra and the category. For any functor $\mathbf{F} : A \to PSet$ we denote by $\mathbf{F}_p$ the value $\mathbf{F}(p)$ of functor $\mathbf{F}$ for object p from A. For any q and p such that $p \leq q$ a functor $\mathbf{F}$ defines the function from $\mathbf{F}_p$ to $\mathbf{F}_q$ which we denote $\mathbf{F}_{pq}$. A functor $\mathbf{F}$ will be treated as the collection $\{\mathbf{F}_p : p \in A\}$ of sets indexed by elements of the set A from an algebra A and endowed with the transition mapping $\mathbf{F}_{pq} : \mathbf{F}_p \to \mathbf{F}_q$ under $p \leq q$ (in particular, $\mathbf{F}_{pp}$ will an identity function on $\mathbf{F}_p$).

We continue in this fashion putting $\mathbf{\Omega}_p = [p)^+$ and for p and q such that $p \leq q$ the function $\mathbf{\Omega}_{pq} : \mathbf{\Omega}_p \to \mathbf{\Omega}_q$ maps every $S \in [p)^+$ into $S \cap [q) \in [q)^+$, i.e. $\mathbf{\Omega}_{pq}(S) = S_q$.

A constant functor $1 : A \to PSet$ which is a terminal object of the category $PSet^A$ might be defined with a help of conditions $1_p = \{0\}$ for $p \in A$ and $1_{pq} = id_{\{0\}}$ under $p \leq q$. A subobject classifier $true : 1 \to \mathbf{\Omega}$

is a natural transformation whose p-th component $true_p : \{0\} \to \mathbf{\Omega}_p$ will be determined by the equality $true_p(0) = [p)$. Thus, the function $true$ chooses the greatest element from every da Costa algebra of $[p)^+$ type.

Let $\tau : \mathbf{F} \underset{\bullet}{\rightarrowtail} G$ be an arbitrary subobject of $PSet^A$-object $\mathbf{G}$. An every component τ_p is injective and can be treated as the inclusion function $\mathbf{F}_p \hookrightarrow \mathbf{G}_p$. The p-th component $(\chi_\tau)_p : \mathbf{G}_p \to [p)^+$ of a characteristic arrow $\chi_\tau : \mathbf{G} \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$ will be defined by the equality

$$(\chi_\tau)_p(x) = \{q : p \leq q \text{ and } \mathbf{G}_{pq}(x) \in \mathbf{F}_q\}$$

for every $x \in \mathbf{G}_p$.

Now we construct truth arrows in a potos $PSet^A$. Let us start with an arrow $false$.

An initial object $0 : A \to PSet$ of category $PSet^A$ is the constant functor such that $0_p = \varnothing^*$ and $0_{pq} = id_{\varnothing*}$ for $p \leq q$. Components of a natural transformation $0 \underset{\bullet}{\rightarrowtail} 1$ are the inclusions $\varnothing^* \hookrightarrow \{0\}$ (the same component for any p). According to the usual definition an arrow $false$ is the characteristic arrow of subobject $! : 0 \underset{\bullet}{\rightarrowtail} 1$. For its component $false_p : \{0\} \to \mathbf{\Omega}_p$ we have $false_p(0) = \{q : p \leq q \text{ and } 1_{pq}(0) \in 0_q\} = \{q : p \leq q \text{ and } 0 \in \varnothing^*\} = \varnothing^*$ and hence a natural transformation chooses the null element from an every da Costa algebra.

Conjunction and disjunction can be handled in same way as in case of topos Set^P (cf. [7]), i.e. we, in fact, need for $\cap : \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{\Omega}$ and $\cup : \mathbf{\Omega} \times \mathbf{\Omega} \to \mathbf{\Omega}$ the definitions of their p-th components in a form of

$$\cap_p(\langle S, T\rangle) = S \cap T;$$
$$\cup_p(\langle S, T\rangle) = S \cup T.$$

The negation is $\neg : \mathbf{\Omega} \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$ whose p-th component $\neg_p : \mathbf{\Omega}_p \to \mathbf{\Omega}_p$ in case of indentifying $false_p$ with the inclusion $\{\varnothing^*\} \hookrightarrow \mathbf{\Omega}_p$ (and since $\neg : \mathbf{\Omega} \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$ is a characteristic arrow of subobject $false$) is as follows:

$$\neg_p(S) = \{q : p \leq q \text{ and } \mathbf{\Omega}_{pq}(S) \in \{\varnothing^*\}\} = \{q : p \leq q \text{ and } S \cap [q) = \varnothing^*\} = [p) \cap \neg S = (\neg S)_p.$$

A negation $\neg^* : \mathbf{\Omega} \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$ is obtained by deducing that the p-th component $\neg^*_p : \mathbf{\Omega}_p \to \mathbf{\Omega}_p$ of negation satisfies equality

$$\neg^*_p(S) = (\neg S)_p \cap_p (\neg S^o)_p = \cap_p(\langle \neg_p(S), \neg_p(S^o)\rangle) = (S')_p.$$

An implication $\Rightarrow : \mathbf{\Omega} \times \mathbf{\Omega} \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$ we have by defining the p-th component as

$\Rightarrow_p (\langle S, T \rangle) = (S \Rightarrow T)_p$.

Finally, we *will call* $PSet^A$-valuation a function $V : \Phi_0 \to PSet^A(1, \mathbf{\Omega})$ assigning to every propositional variable π_i some truth-value $V(\pi_i) : 1 \underset{\bullet}{\rightarrowtail} \mathbf{\Omega}$. This function might be extended to the set Φ of all formulas in the following way:

(12) $V(\alpha) = false \Rightarrow V(\neg\alpha) = true$:

(13) $V(\neg\neg\alpha) = true \Rightarrow V(\alpha) = true$;

(14) $V(\beta^\circ) = V(\alpha \supset \beta) = V(\alpha \supset \neg\beta) = true \Rightarrow V'(\beta^\circ) = V'(\alpha \supset \beta) = V'(\alpha \supset \neg\beta) = 1$;

(15) $V(\alpha \supset \beta) = true \Leftrightarrow V(\alpha) = false$ or $V(\beta) = true$;

(16) $V(\alpha \wedge \beta) = true \Leftrightarrow V(\alpha) = V(\beta) = true$;

(17) $V(\alpha \vee \beta) = true \Leftrightarrow V(\alpha) = 1$ or $V(\beta) = true$;

(18) $V(\alpha^\circ) = V(\beta^\circ) = true \Rightarrow V'(\alpha^\circ) = V'(\beta^\circ) = 1$.

We say that the formula α be $PSet^A$-valid (we write $PSet^A \models \alpha$) if $V(\alpha) = true : 1 \to \mathbf{\Omega}$ for all $PSet^A$-valuations V.

Using da Costa-Alves valuation $V' : \Phi_0 \to \{0, 1\}$ from above it is easy to prove at the same way the following theorem:

THEOREM 8. *For any potos $PSet^A$, $PSet^A \models \alpha$ iff $\vdash_{C_1} \alpha$ (i.e. α is provable in C_1).*

References

[1] Benabou, J. "Fibered Categories and the Foundations of Naive Category Theory", *The Journal of Symbolic Logic*, 1983, Vol. 50, No. 4, pp. 9–37.

[2] Caleiro, C., Gonçalves, R. "Behavioral algebraization of da Costa's C-systems", *Journal of Applied Non-Classical Logics*, 2009, Vol. 19, No. 2, pp. 127–148.

[3] Carnielli, W.A., Alcantara, L.P. "Paraconsistent algebras", *Studia Logica*, 1984, Vol. XLIII, No. 1/2, pp. 79–87.

[4] da Costa, N.C.A. "Calculus propositionnels pour les systemes inconsistants", *C.R. Acad. Sci. Paris*, 1963, T. 257, pp. 3790–3792.

[5] da Costa, N.C.A. "Paraconsistent Mathematics", in: *Frontiers of Paraconsistent Logic*, D. Batens et al. (eds.). Research Studies Press Ltd., Baldock, Hartfordshire, England, 2000, pp. 166–179.

[6] da Costa, N.C.A., Alves, E.H. "A Semantical Analyses of the Calculi C_n", *Notre Dame Journal of Formal Logic*, 1977, Vol. XVIII, No. 4, pp. 621–630.

[7] Goldblatt, R. *Toposes. The categorial analysis of logic.* Amsterdam, North Holland, 1973.

[8] Grishin, V.N. "Weight of the comprehension axiom in a theory based on logic without contractions", *Mathematical Notes*, 1999, Vol. 66, No. 5, pp. 533–540.

[9] Shepherdson, J.C. "Inner models for set theory", *Journal of Symbolic Logic*, 1951, Vol. 15, pp. 161–190.

[10] Vasyukov, V.L. "Paraconsistency in Categories", in: *Frontiers of Paraconsistent Logic*, D. Batens, C. Mortensen, G. Priest and J.-P. van Bendegem (eds.). Research Studies Press Ltd., Baldock, Hartfordshire, England, 2000, pp. 263–278.

[11] Vasyukov, V.L. Potosy dlya paraneprotivorechivoi logiki [Potoses for Paraconsistent Logics], Modern Logics: Issues of Theory, History and Implementation in Science. Papers from the X-th all-Russian Scientific Conference, SPb, 2008. P. 105–107. (In Russian)