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*Неклассическая логика*  
*Non-classical Logic*

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## Quantum categories for quantum logic

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**Abstract:** The paper is the contribution to quantum toposophy focusing on the abstract orthomodular structures (following Dunn-Moss-Wang terminology). Early quantum toposophical approach to “abstract quantum logic” was proposed based on the topos of functors  $[E, \mathbf{Sets}]$  where  $E$  is a so-called orthomodular preorder category — a modification of categorically rewritten orthomodular lattice (taking into account that like any lattice it will be a finite co-complete preorder category). In the paper another kind of categorical semantics of quantum logic is discussed which is based on the modification of the topos construction itself — so called *quantos* — which would be evaluated as a non-classical modification of topos with some extra structure allowing to take into consideration the peculiarity of negation in orthomodular quantum logic. The algebra of subobjects of quantos is not the Heyting algebra but an orthomodular lattice. Quantoses might be apprehended as an abstract reflection of Landsman’s proposal of “Bohrification”, i.e., the mathematical interpretation of Bohr’s classical concepts by commutative  $C^*$ -algebras, which in turn are studied in their quantum habitat of noncommutative  $C^*$ -algebras — more fundamental structures than commutative  $C^*$ -algebras. The Bohrification suggests that topos-theoretic approach also should be modified. Since topos by its nature is an intuitionistic construction then Bohrification in abstract case should be transformed in an application of categorical structure based on an orthomodular lattice which is more general construction than Heyting algebra — orthomodular lattices are non-distributive while Heyting algebras are distributive ones. Toposes thus should be studied in their quantum habitat of “orthomodular” categories i.e. of quantoses. Also an interpretation of some well-known systems of orthomodular quantum logic in quantos of functors  $[E, \mathbf{QSets}]$  is constructed where  $\mathbf{QSets}$  is a quantos (not a topos) of quantum sets. The completeness of those systems in respect to the semantics proposed is proved.

**Keywords:** Quantum logic, quantum conditional, quantos, polynomial exponentiation, quantum sets

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## 1. Introduction

If following M. Dunn, L. Moss and Z. Wang [Dunn et al., 2013] we divide the history of quantum logic into three eras or “lives” then the first life featuring Birkhoff and von Neumann’s algebraic approach in the 1930’s should be called “concrete quantum logic” [Birkhoff, Neumann, 1936]. It was the period of the atomic, non-distributive, orthomodular Hilbert lattice of projections on an infinite dimensional Hilbert space. The second life that began in the late 1950’s and blossomed in the 1970’s should be called “abstract quantum logic” because focusing on the abstract orthomodular structures. And the third life of recent developments in quantum logic coming from its connections to quantum computation and should be called “computational quantum logic”.

But this qualification is not accurate since in modern literature we can find investigations dealing both with category theory and quantum logic that impell us to speak of one more life of quantum logic — “categorical quantum logic”. Here the subject may be the categorical (topos) foundations for theories of physics (cf. e.g. [Isham, Doring, 2007]), sometimes it is the logic of strongly compact closed categories with biproducts [Abramsky, Duncan, 2004], the category of sheaves over a quantaloid (a quantum topos) (cf. [Crane, 2007]) etc. In case of “topos quantum logic” (*quantum toposophy*) we face with the exploitation both “concrete” and “abstract” quantum logic approaches.

In particular, in [Landsman, 2017, p.461] the topos under consideration is a topos  $\mathbf{T}(A)$  of functors  $F: \mathfrak{C}(A) \rightarrow \mathbf{Sets}$ , i.e.,  $\mathbf{T}(A) = [\mathfrak{C}(A), \mathbf{Sets}]$  where  $A$  is a unital  $C^*$ -algebra (in  $\mathbf{Sets}$ ), with associated poset  $\mathfrak{C}(A)$  of all unital commutative  $C^*$ -subalgebras  $C \subset A$  ordered by inclusion.  $\mathfrak{C}(A)$  is regarded as a (posetal) category, in which there is a unique arrow  $C \rightarrow D$  iff  $C \subseteq D$  and there are no other arrows. Since for any poset  $X$  we have an isomorphism of categories  $[X, \mathbf{Sets}] \simeq \mathbf{Sh}(X)$ , then we obtain  $\mathbf{T}(A) \simeq \mathbf{Sh}(\mathfrak{C}(A))$ . Moreover, for any small category  $\mathbf{C}$  an internal  $C^*$ -algebra in the associated presheaf topos  $[\mathbf{C}^{op}, \mathbf{Sets}]$  is given by a contravariant functor  $\underline{A}: \mathbf{C} \rightarrow \mathbf{CA}$ , where  $\mathbf{CA}$  is the category that has  $C^*$ -algebras as objects and homomorphisms as arrows (but this is not true for internal  $C^*$ -algebras on sheaf toposes  $\mathbf{T} = \mathbf{Sh}(X)$ ).

More abstract approach is proposed in [Vasyukov, 1989], [Vasyukov, 2005] by means of the topos of functors  $[\mathbf{E}, \mathbf{Sets}]$  where  $\mathbf{E}$  is a so-called *orthomodular preorder category* — modification of a categorically rewritten orthomodular lattice (taking into account that like any lattice it will be a finite co-complete preorder category). In fact, this is a lattice preorder category with the functor rendering the orthocomplementation properties. More formally:

**Definition 1.** An ortho preorder category  $\mathbf{E}$  is a preorder category equipped with the contravariant functor  $\mathbb{N}: \mathbf{E} \rightarrow \mathbf{E}$  such that

- (i)  $\mathbf{E}$  has an initial object  $0$  and a terminal object  $1$ ;
- (ii)  $\mathbf{E}$  has finite products  $\langle -, - \rangle$  and finite co-products  $[-, -]$ ;
- (iii)  $\underline{N}^2 a \cong a$  for any object  $a$  in  $\mathbf{E}$ ;
- (iv)  $\langle a, \underline{N}a \rangle \simeq 0$ ,  $[a, \underline{N}a] \simeq 1$  for any object  $a$  in  $\mathbf{E}$ ;
- (v)  $\underline{N}[a, b] \simeq \langle \underline{N}a, \underline{N}b \rangle$ ,  $\underline{N}\langle a, b \rangle \simeq [\underline{N}a, \underline{N}b]$  for any two objects  $a, b$  in  $\mathbf{E}$ .

An ortho preorder category  $\mathbf{E}$  is an orthomodular preorder category when in addition the following condition is satisfied:

- (vi) if  $a \rightarrow b$  is an arrow in  $\mathbf{E}$  then  $[a, \langle \underline{N}a, b \rangle] \simeq b$  for any two objects  $a, b$  in  $\mathbf{E}$ .

It is instructive that K. Landsman in his book “Foundations of Quantum Theory” [Landsman, 2017] writes that the topos-theoretic approach to quantum mechanics from his point of view encompasses quantum logic. But he at once remarks that if one adheres to the doctrine of classical concepts, then quantum logic turns out to be intuitionistic and hence distributive, rather than orthomodular. This is tightly connected with Bohr’s doctrine of classical concepts that in the systems to which the quantum mechanical formalism is to be applied their quantum mechanical treatment will for this purpose be essentially equivalent with a classical description.

Landsman’s proposal of “Bohrification”, i.e., the mathematical interpretation of Bohr’s classical concepts by commutative  $C^*$ -algebras, which in turn are studied in their quantum habitat of noncommutative  $C^*$ -algebras, suggests that topos-theoretic approach also should be modified. Noncommutative  $C^*$ -algebras are more fundamental structures than commutative  $C^*$ -algebras. The same concerns orthomodular lattices which are more general than Heyting algebras underlined intuitionistic logic — orthomodular lattices are non-distributive while Heyting algebras are distributive ones. Since topos by its origin is an intuitionistic construction then “Bohrification” in this case should be transformed in an application of categorical structure based on orthomodular lattice. Toposes thus should be studied in their quantum habitat of “orthomodular” categories.

The construction of the topos  $[\mathbf{E}, \mathbf{Sets}]$  where  $\mathbf{E}$  is a categorically treated orthomodular lattice from [Vasyukov, 2005] would be regarded, in a sense, as an embedding of quantum logic (based on orthomodular lattice) into intuitionistic universe since  $\mathbf{Sets}$  is a topos. Does we always need an intuitionistic universe as the basis of quantum logic considerations? More natural seems the exploitation of categories having quantum logic as its inner structure. There are formulations of quantum set theories (cf., e.g. [Takeuti, 1981]) which differ from

classical  $ZF$  set theory and this allows one to think that such sets does not generate a topos. In this case  $[E, \mathbf{QSets}]$  (where  $\mathbf{QSets}$  is a category of quantum sets) does not be a topos and the interpretation above fails.

Following this line the construction of *quantos* is introduced which would be evaluated as a non-classical modification of topos with some extra structure allowing to treat the peculiarity of negation in quantum logic. The category  $\mathbf{QSets}$  turns out to be a quantos and it is proved that the category  $[E, \mathbf{QSets}]$  be a quantos too. The systems of quantum logic are intrpreted in both types of quantoses in more natural way then it was done in case of  $[E, \mathbf{Sets}]$ . In a sense, quantoses should be considered as a quantum categorical universe for quantum logic considerations.

The well-known Goldblatt's, Nishimura's and Cutlend-Gibbins' systems of quantum logic are interpreted in a topos  $[E, \mathbf{QSets}]$ . This interpretation is extended to the system of quantum logic by G. Hardegree with Sasaki arrow playing the role of quantum conditional.

## 2. Quantoses

To interpret quantum logic in more natural way we will introduce a special kind of non-standard categories more suitable for interpretation of quantum logics because of their "orthomodular" structure.

A *quantos* should be considered as a topos equipped with some additional structure. One might equally exploit the name "quantum topos" or "orthomodular topos" which are more informative on the peculiarities of its inner structure.

**Definition 2.** A quantos  $\mathbf{Q}$  is a topos which is also complementary closed and orthomodular one.

Complementary closedness here means that

- (i) for any object  $a$  of  $\mathbf{Q}$  there is an object  $a'$  such that
  - for any arrow  $f : a \rightarrow d$  of  $\mathbf{Q}$  we have a mono  $f' : a' \rightarrow d$ ,  $a'' \cong a$ ,
  - for any arrow  $g : a \rightarrow b$  of  $\mathbf{Q}$  there is an arrow  $g : b' \rightarrow a'$ ;
- (ii)  $a + a' \cong 0, a \times a' \cong 1$  (with, possibly, binary coproducts and products) for any object  $a$  in  $\mathbf{Q}$ ;
- (iii)  $(a + b)' \cong a' \times b', (a \times b)' \cong a' + b'$  for any two objects  $a, b$  in  $\mathbf{Q}$ .

*Othomodularity means that*

- (iv) for any pair of objects  $a, b$  of  $\mathbf{Q}$  there are objects

$$a \supset_1 b \cong (a' \times b) + (a' \times b') + (a \times (a' + b))$$

$$a \supset_2 b \cong (a' \times b) + (a \times b) + ((a' + b) \times b')$$

$$a \supset_3 b \cong a' + (a \times b)$$

$$a \supset_4 b \cong b + (a' \times b')$$

$$a \supset_5 b \cong (a' \times b) + (a \times b) + (a' \times b')$$

such that

$$a \supset_i b \cong b^a \quad (1 \leq i \leq 5).$$

The last point gives us a “polynomial exponentiation” in  $\mathbf{Q}$  because we will have five evaluation arrows  $ev : a \supset_i b \times a \rightarrow b$  ( $1 \leq i \leq 5$ ).

**Proposition 1.** *In quantos  $\mathbf{Q}$  the collection  $Sub(d)$  of all  $\mathbf{Q}$ -arrows that are monic with  $d$  as codomain is an orthomodular lattice.*

**Proof.** Since any quantos  $\mathbf{Q}$  is a topos then for any object  $d$  in  $\mathbf{Q}$  the collection  $Sub(d)$  of all  $\mathbf{Q}$ -arrows that are monic with  $d$  as codomain will be preordered bounded lattice. The points (i)–(iii) of the definition 2 transform this collection into ortholattice (cf. [Dalla Chiara, Giuntini, 2002, p. 137]) and point (iv) leads to  $Sub(d)$  be an orthomodular lattice (cf. [Dalla Chiara, Giuntini, 2002, p. 142]). ■

It is easily can be seen that in quantos we have  $Sub(d) \cong Hom(d, \Omega)$  and thus  $Hom(d, \Omega)$  will be an orthomodular lattice. But in this case the problem arises concerning the category **Sets**. The matter of fact is that in **Sets** we have  $Sub(D) \cong \mathcal{P}(D)$  where  $\mathcal{P}(D) = \{x : x \text{ is a subset of the set } D\}$ . Since  $\mathcal{P}(D)$  is a Boolean algebra of subsets and not the orthomodular lattice of sets, then we come to the conclusion that *Set* cannot be a quantos. But according to [Dalla Chiara, Giuntini, 2002, p. 144] (using the dual version of the theorem) or lemma 2.1.1 from [Vasyukov, 2005, p. 39] a lattice  $E^+ = (E^+, \subseteq, *)$  of \*-closed quasi-hereditary sets is an orthomodular lattice of sets and thus there are some sets which form such an algebra. So, either such sets generates the subcategory **QSets** of **Sets** or **Sets** is, in a sense, a subcategory of **QSets**.

G.Takeuti [Takeuti, 1981] in 1981 has been developed an important application of quantum logic to set theory. He constructed an orthomodular-valued model for set theory where the set of truth-values is supposed to have the algebraic structure of a complete orthomodular lattice instead of complete Boolean algebras in case of the usual Boolean-valued models. The standard axioms of set theory hold in orthomodular-valued model only in restricted form. Since the collection of all sets, plus  $\emptyset, V, \cap, \cup, C$  (where  $V = \{x : x = x\}$  and

$x^C = \{y : y \notin x\}$ ) form in  $ZF$  a complete Boolean algebra then it is easy to check that an algebra of sets in Takeuti's quantum set theory will be an orthomodular lattice. Then it seems attractive to consider category **QSets** as a category whose objects are exactly sets from an orthomodular-valued model.

**Proposition 2.** *QSets is a quantos.*

**Proof.**[Sketch of proof] For any set  $I$  we will have an orthomodular lattice of  $*$ -closed quasi-hereditary subsets  $(I^+, \subseteq, *, \emptyset, [I])$  where  $I^+ \subseteq \mathcal{P}(I)$ . If we consider inclusion functions as arrows then we can define  $[x] \leq [y]$  iff  $[x] \hookrightarrow [y]$  and put  $[x]' = [x]^*$  for complementary closedness. Thus, we can conclude that in **QSets** we have  $Sub(d) \cong \mathcal{P}(d)$  and  $Sub(d)$  will be an orthomodular lattice and so do  $\mathcal{P}(d)$ . But in this case we cannot take 2 as the classifying object exploiting the property that  $\mathcal{P}(d) \cong 2^d$  because this gives rise to the Boolean algebra of characteristic arrows as in **Sets**. Moreover, we need to take into account that we deal with the orthomodular-valued sets in the universe  $V^E$  which is constructed as  $V^E = \bigcup_{\nu \in On} V(\nu)$ , where

$$V(0) = \emptyset;$$

$$V(\nu + 1) = \{g : g \text{ is a function and } Dom(g) \subseteq V(\nu) \text{ and } Rang(g) \subseteq E\};$$

$$V(\lambda) = \bigcup_{\nu < \lambda} V(\nu), \text{ for any limit-ordinal } \lambda.$$

( $Dom(g)$  and  $Rang(g)$  are the *domain* and the *range* of function  $g$ , respectively)

It means that given an orthomodular universe  $V^E$  for any formula  $\alpha$  we should define the truth-value  $\|\alpha\|^\sigma$  in a complete orthomodular lattice  $E$  as induced by any interpretation  $\sigma$  of the variables in the universe  $V^E$ . Hence, for any  $y \subseteq d$  unlike the usual definition of the character-

istic arrow  $\chi_y(x) = \begin{cases} 1, & \text{if } x \in y \\ 0, & \text{if } x \notin y \end{cases}$  in **Sets** we have  $\chi_y(x) = \|x \in y\|^\sigma =$

$\bigvee_{g \in Dom(\sigma(y))} \{\sigma(y)(g) \wedge \|x = z\|^{\sigma[z/g]}\}$  in **QSets** (cf. [Dalla Chiara, Giuntini, 2002, p. 177]). This means that as the classifying object in **QSets** we should take not the two-element Boolean algebra but an orthomodular lattice  $E$ .

So, we have  $\mathcal{P}(d) \cong E$  together with the function  $true : 1 \rightarrow E$  (such that  $true(\emptyset) = 1$ ) playing the role of the subobject classifier in **QSets**. Also we have respectively an arrow  $false : 1 \rightarrow E$  (such that  $false(\emptyset) = \emptyset$ ). ■

Now we define truth-arrows in quantos in general case. Let us **Q** will be a quantos with the subobject classifier  $true : 1 \rightarrow \Omega$ . Then the negation  $\neg : \Omega \rightarrow \Omega$  will be the unique arrow for which the diagram

$$\begin{array}{ccc}
1 & \xrightarrow{\text{false}} & \Omega \\
\downarrow & & \downarrow \neg \\
1 & \xrightarrow{\text{true}} & \Omega
\end{array}$$

will be the pullback in  $\mathbf{Q}$ . Thus,  $\neg = \chi_{\text{false}}$ . Given  $f : a \multimap d$ , the orthocomplement of  $f$  (relative to  $d$ ) is the subobject  $f' : a' \multimap d$  whose character is  $\neg \circ \chi_f$ . Thus  $f'$  is defined to be the pullback

$$\begin{array}{ccc}
a' & \xrightarrow{f'} & d \\
\downarrow & & \downarrow \neg \circ \chi_f \\
1 & \xrightarrow{\text{true}} & \Omega
\end{array}$$

of  $\text{true}$  along to the  $\neg \circ \chi_f$ , yielding  $\chi_{f'} = \neg \circ \chi_f$ , by definition.

Since quantos is a topos then conjunction truth-arrow will be defined standardly:

$\cap : \Omega \times \Omega \rightarrow \Omega$  is a character of the product of arrows  $\langle \text{true}, \text{true} \rangle : 1 \rightarrow \Omega \times \Omega$  in a quantos  $\mathbf{Q}$ .

As to the disjunction arrow then it is not a primitive and should be defined by means of negation and conjunction arrows.

Finally, we can define conditional taking  $\supset : \Omega \times \Omega \rightarrow \Omega$  in the usual way as a character of the monic  $e : \otimes \multimap \Omega \times \Omega$ , which is an equalizer of the pair

$$\Omega \times \Omega \begin{array}{c} \xrightarrow{\cap} \\ \xrightarrow{pr_1} \end{array} \Omega$$

where  $pr_1$  is a projection on the first member of the product  $\Omega \times \Omega$ . But since our conditional is defined via polynomial exponentials then we, as consequence, will have not one but five conditionals.

### 3. Interpretation of Quantum Logic in quantoses

R. Goldblatt in his paper *Semantic analysis of orthologic* [Goldblatt, 1974] treats logics as not a set of well-formed formulas but as the collection of their ordered pairs satisfying certain closedness condition. Logics of such a type he calls binary ones. They are characterized by the class of ortho-, orthomodular lattices in terms of  $A \vdash B$  iff  $v(A) \leq v(B)$  where  $v$  is a function from the set of well-ordered formulas into an ortholattice in which connectives  $\neg$  and  $\wedge$  are interpreted as an orthocomplementation and a lattice meet respectively. His

system  $O$  of orthologic characterized by the class of ortholattice is defined by means of the following axiomatics:

$$\begin{array}{ll}
 \mathbf{Axioms.} & (1) \alpha \vdash \alpha \qquad (2) \alpha \wedge \beta \vdash \alpha \\
 & (3) \alpha \wedge \beta \vdash \beta \qquad (4) \alpha \vdash \neg\neg\alpha \\
 & (5) \neg\neg\alpha \vdash \alpha \qquad (6) \alpha \wedge \neg\alpha \vdash \beta \\
 \\
 \mathbf{Rules.} & (7) \frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma} \qquad (8) \frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \vdash \beta \wedge \gamma} \\
 & (9) \frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}
 \end{array}$$

Here  $\alpha \vdash \beta$  means informally that  $\beta$  can be inferred from  $\alpha$ . This notation can be extended to  $\Gamma \vdash \alpha$  where  $\Gamma$  is a set of well-formed formulas and putting that  $\Gamma \vdash \alpha$  iff for some  $\beta_1, \beta_2, \dots, \beta_n \in \Gamma$  we have  $\beta_1 \wedge \beta_2 \wedge \dots \wedge \beta_n \vdash \alpha$ .

One can pass from an orthologic  $O$  to quantum orthologic  $OM$  which is characterized by the class of orthomodular lattices while employing the definition  $\alpha \vee \beta =_{def} \neg(\neg\alpha \wedge \neg\beta)$  and adding to  $O$  one more axiom

$$(10) \alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash \beta$$

Regarding Goldblatt's binary relation as an analogue of Gentzen's natural deduction H. Nishimura [Nishimura, 1980] elaborated sequential system  $GO$  for orthologic and  $GOM$  for quantum logics with  $\wedge$  and  $\neg$  as the primitive connectives. His formulation of  $GOM$  is as follows:

**Axioms.**  $\alpha \vdash \alpha$

**Rules.**

$$\begin{array}{ll}
 \frac{\Gamma \rightarrow \Delta}{\Pi, \Gamma \rightarrow \Delta, \Sigma} \text{ (extension)} & \frac{\Gamma_1 \rightarrow \Delta_1, \alpha \quad \alpha, \Gamma_2 \rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2} \text{ (cut)} \\
 \\
 \frac{\alpha, \Gamma \rightarrow \Delta}{\alpha \wedge \beta, \Gamma \rightarrow \Delta} (\wedge \rightarrow) & \frac{\beta, \Gamma \rightarrow \Delta}{\alpha \wedge \beta, \Gamma \rightarrow \Delta} (\wedge \rightarrow) \\
 \\
 \frac{\Gamma \rightarrow \Delta, \alpha \quad \Gamma \rightarrow \Delta, \beta}{\Gamma \rightarrow \Delta, \alpha \wedge \beta} (\rightarrow \wedge) & \\
 \\
 \frac{\Gamma \rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \rightarrow \Delta} (\neg \rightarrow) & \frac{\Gamma \rightarrow \Delta}{\neg\Delta \rightarrow \neg\Gamma} (\rightarrow \neg) \\
 \\
 \frac{\alpha, \Gamma \rightarrow \Delta}{\neg\neg\alpha, \Gamma \rightarrow \Delta} (\neg\neg \rightarrow) & \frac{\Gamma \rightarrow \Delta, \alpha}{\Gamma \rightarrow \Delta, \neg\neg\alpha} (\rightarrow \neg\neg)
 \end{array}$$



We come to the sequential system *GOM* by adding to *GO* the following rule:

$$\frac{\neg\beta \rightarrow \neg\alpha \quad \alpha, \beta \rightarrow}{\neg\alpha \rightarrow \neg\beta} \text{ (OM)}$$

The cut-elimination theorem fails for *GO* and *GOM* but the following claim is true: if the sequent  $\Gamma \rightarrow \Delta$  (where  $\Delta$  is not empty) is provable then for some  $\alpha \in \Delta$ ,  $\Gamma \rightarrow \alpha$  is provable too. This claim leads, in particular, to the normalization theorem which is proved for both systems.

Unfortunately, it is known that Nishimura's calculus have two defects:

1. in quantum logic connectives  $\wedge$  and  $\vee$  usually are dual ones while this is not true for Nishimura's calculus (sequential calculus is dual iff for all finite  $\Gamma$  and  $\Delta$  we always obtain  $\vdash \Gamma \rightarrow \Delta$  iff  $\vdash \Gamma^* \rightarrow \Delta^*$  where for  $\alpha$  we obtain  $\alpha^*$  by replacing  $\wedge$  with  $\vee$  and the other way round; for a set  $\Gamma$  of formulas we obtain dual set  $\Gamma^* = \{\gamma^* : \gamma \in \Gamma\}$  but we need to take into account that such replacing in case of quantum logics supposes the choice of just one of the connectives  $\wedge, \vee$  as a primitive).
2. this calculi is non-regular in that sense that the condition  $\Gamma_1, \dots, \Gamma_n \rightarrow \Delta_1, \dots, \Delta_n$  iff  $\Gamma_1 \wedge \dots \wedge \Gamma_n \rightarrow \Delta_1 \vee \dots \vee \Delta_n$  fails.

N.J. Cutland and P.F. Gibbins [Cutland, Gibbins, 1982] proposed a regular sequent calculus of quantum logic which is free of those shortcomings but in which unlike Nishimura's calculus the usual cut- rule is not accepted.

Axiomatics of their system *GO*<sup>†</sup> which is an extension of Nishimura's system *GO* is as follows:

**Axioms.**  $\alpha \vdash \alpha$

**Rules.**

$$\frac{\Gamma \rightarrow \Delta}{\Theta, \Gamma \rightarrow \Delta, \Sigma} \text{ (extension)}$$

$$\frac{\Gamma \rightarrow \alpha, \Delta_1 \quad \alpha \rightarrow \Delta_2}{\Gamma \rightarrow \Delta_1, \Delta_2} \text{ (cut-1)} \quad \frac{\Gamma_1 \rightarrow \alpha \quad \Gamma_2, \alpha \rightarrow \Delta}{\Gamma_1, \Gamma_2 \rightarrow \Delta} \text{ (cut-2)}$$

$$\frac{\alpha, \Gamma \rightarrow \Delta}{\alpha \wedge \beta, \Gamma \rightarrow \Delta} (\wedge \rightarrow) \quad \frac{\beta, \Gamma \rightarrow \Delta}{\alpha \wedge \beta, \Gamma \rightarrow \Delta} (\wedge \rightarrow)$$

$$\frac{\Gamma \rightarrow \alpha \quad \Gamma \rightarrow \beta}{\Gamma \rightarrow \alpha \wedge \beta} (\rightarrow \wedge)^\dagger \quad \frac{\alpha \rightarrow \Delta \quad \beta \rightarrow \Delta}{\alpha \vee \beta \rightarrow \Delta} (\vee \rightarrow)^\dagger$$

$$\begin{array}{ccc}
\frac{\Gamma \rightarrow \Delta, \alpha}{\Gamma \rightarrow \Delta, \alpha \vee \beta} (\rightarrow \vee)^\dagger & & \frac{\Gamma \rightarrow \Delta, \beta}{\Gamma \rightarrow \Delta, \alpha \vee \beta} (\rightarrow \vee)^\dagger \\
\frac{\Gamma \rightarrow \alpha}{\Gamma, \neg\alpha \rightarrow} (\neg \rightarrow)^\dagger & & \frac{\Gamma \rightarrow \Delta}{\neg\Delta \rightarrow \neg\Gamma} (\rightarrow \neg) \\
\frac{\alpha, \Gamma \rightarrow \Delta}{\neg\neg\alpha, \Gamma \rightarrow \Delta} (\neg\neg \rightarrow) & & \frac{\Gamma \rightarrow \Delta, \alpha}{\Gamma \rightarrow \Delta, \neg\neg\alpha} (\rightarrow \neg\neg)
\end{array}$$

Rules with the sign  $\dagger$  are specific rules of  $GO^\dagger$ . We obtain the system of quantum logic  $GO^\dagger M$  by adding Nishimura's rule of orthomodularity, that is  $GO^\dagger M = GO^\dagger + (OM)$ .

To describe an algebraic semantic of all those calculi we need to introduce some concepts.

Firstly consider the concept of *hereditary sets* in an orthomodular lattice  $E$ . For any element  $p$  the hereditary set  $[p]$  is defined by the equality

$$[p] = \{q : p \leq q\}.$$

An orthocomplementation  $^\perp$  in  $E$  is an involutive permutation where  $b^\perp \leq a^\perp$  whenever  $a \leq b$  ( $a, b \in E$ ). It is known (cf. [Birkhoff, 1967]) that in orthomodular lattices an every interval  $[a, b]$  is an orthomodular lattice closed under  $\wedge, \vee$  and relative complementation  $c' = (a \vee c^\perp) \wedge b = a \vee (c^\perp) \wedge b$ . For the hereditary sets an upper limit of interval is 1 and therefore  $c' = (p \vee c^\perp) \wedge 1 = p \vee c^\perp$ . Hence, the set  $E^+$  of all hereditary sets will be the set of orthomodular lattices.

To transform the lattice  $E^+ = (E^+, \subseteq)$  into an orthomodular lattice we need to define an orthocomplementation. In fact, this procedure should specify an involutive operation on  $E^+$ . From the definition of orthocomplementation it follows that if  $c' = p \vee c^\perp$  then  $c' \in [p]$ . Thus it seems natural to define  $[p]'$  as a set of such  $c$  that  $c' \in [p]$ . In this case  $p \leq c^\perp$  but this is exactly the definition of an orthogonality relation since it is specified by the request that  $a \perp b$  whenever  $a \leq b^\perp$ . It is known that the relation of orthogonality is symmetric and irreflexive.

Now we define  $x \perp Y$  iff for any  $y \in Y$ ,  $x \perp y$  and then introduce an operation  $*$  by means of the definition:

$$(i) [p]^* = \{x : x \perp [p]\}.$$

A set  $X$  is to be called *closed relative to  $*$*  if  $(X^*)^* = X$ .

But from the definition (i) it follows that  $[p]^* = \emptyset$  because  $1 \in [p]$  and  $x \perp 1$  if  $x \leq 0$  i.e.  $x = 0$ . To avoid this let us modify the definition of hereditary sets in the following way:  $[p] = \{q : p \leq q \ \& \ q \neq 1\}$ .

Such sets usually are called *quasi-hereditary* ones but in order not to overburden the terminology we retain the original term (of hereditary sets). It is easy to reformulate all previous definitions taking into consideration the limitation accepted.

**Lemma 1.** *A lattice  $E^+ = (E^+, \subseteq, *)$  of  $*$ -closed hereditary sets is an orthomodular lattice.*

**Proof.** A partially ordered by inclusion set of hereditary sets is a bounded distributive lattice whose meets and joins are specified by respective set-theoretical operations  $\cap$  and  $\cup$ . Hence,  $(E^+, \subseteq, *)$  will be a lattice with respect to  $\cap$  and  $\cup$ . Then in virtue of closedness under  $*$ , symmetry and irreflexivity of the orthogonality relation  $\perp$ ,  $[p] \rightarrow [p]^*$  will be an involution while  $(E^+, \subseteq, *)$  be an ortholattice (cf. [Birkhoff, 1967]). Since an every distributive lattice is a modular one and every modular ortholattice is an orthomodular one then  $(E^+, \subseteq, *)$  will be an orthomodular lattice too.

Observe, that resulted ortholattice will be, in fact, a Boolean algebra. But we can also define  $E^+$  as a non-distributive orthomodular lattice. To that end we can take advantage of the following definition:

$$X \sqcup Y = (X^* \cap Y^*)^*.$$

It is known that in general case  $(X^* \cap Y^*)^* > X \cup Y$ . It is easy to see that  $(E^+, \subseteq, \sqcup, \cap, *)$  is an ortholattice. A necessary and sufficient condition for orthomodularity of  $E^+$  is the following: if  $[x] \subseteq [y]$  and  $[x]^* \cap [y] = \emptyset$  then  $[x] = [y]$ . The proof of the satisfiability of dual condition in  $E^+$  would be found in [Beran, 1984, p. 171]. In fact, we have been obtained a construction which is dual to Janovitz's embedding [Beran, 1984, p. 173]. ■

Note, that in proof of the lemma 5 in fact two lattices  $E_1^+$  and  $E_2^+$  are figured, the former of which is a distributive while the latter is a non-distributive one. In the sequel we will be mean by  $E^+$  the latter.

**Lemma 2.** *A lattice  $[p]^+$  of all  $*$ -closed hereditary sets in  $[p]$  is an orthomodular lattice.*

**Proof.** It is sufficient to put  $[c]_p^* = [c]^* \cap [p]$ . Then in respect to that operation an interval  $[\emptyset, [p]]$  will be an orthomodular lattice (cf. [Birkhoff, 1967]). ■

The relational semantics of all calculi above is described by means of the notions of ortho-, quantum frames and models.

**Definition 3.** An orthoframe is a pair  $\langle X, \perp \rangle$ , where

1.  $X$  is a non-empty set,
2.  $\perp$  is an orthogonality relation on  $X$ .

**Definition 4.** An orthomodel is a triple  $\langle X, \perp, v \rangle$ , where

1.  $\langle X, \perp \rangle$  is an orthoframe,
2.  $v$  is a function assigning to each propositional variable  $p$  a  $*$ -closed subset  $v(\alpha) \subseteq X$ .

**Definition 5.** A quantum frame is a triple  $\langle X, \perp, \Psi \rangle$ , where

1.  $\langle X, \perp \rangle$  is an orthoframe,
2.  $\Psi$  is a non-empty collection of  $*$ -closed subsets of  $X$  such that
  - (a)  $\Psi$  is closed under set intersection and the operation  $*$ ,
  - (b) for any  $Y, Z \in \Psi$ ,  $Y \subseteq Z$  and  $Y * \cap Z = \emptyset$  implies  $Y = Z$ .

**Definition 6.** A quantum model is a 4-tuple  $\langle X, \perp, \Psi, v \rangle$  where

1.  $\langle X, \perp, \Psi \rangle$  is a quantum frame,
2.  $v$  is a function assigning to each propositional variable  $p$  a  $*$ -closed subset  $v(\alpha)$  from  $\Psi$ .

It is easily to be seen that in role of the collection of  $*$ -closed subsets of  $X$  in an orthomodular lattice  $E^+$  would be chosen especially since the condition (b) from the definition 9 is satisfiable in  $E^+$  (this follows from the fact that in ortholattices  $a \leq b$  &  $a^\perp \wedge b = 0 \Rightarrow a = b$  is the necessary and sufficient condition of orthomodularity (cf. [Birkhoff, 1967])).

The quantum model should be defined as a model  $M = \langle E^+, v \rangle$  with the quantum frame  $E^+$  (here  $E^+$  substitutes for the notation  $\langle E, \perp, E^+ \rangle$ ) where  $v : \Phi \rightarrow E^+$  is some  $E$ -valuation and  $\Phi$  is a set of propositional formulas. A valuation  $v : \Phi_0 \rightarrow E$  of the system  $OM$  in orthomodular lattice  $E$  assigning to an every propositional letter  $\pi_i$  some truth-value  $V(\pi_i) \in E$ . It uniquely would be extended in a following way

- (1)  $v(\neg\alpha) = v(\alpha)^\perp$ ;
- (2)  $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$ ;

to the function  $v : \Phi \rightarrow E$ . The sentence  $\alpha$  such that  $v(\alpha) = 1$  for every  $E$ -valuation  $v$  is called  $E$ -valid and this is denoted as  $E \models \alpha$ .

**Theorem 1.** *For any orthomodular lattice  $E$  we have  $\vdash_{OM, GOM, GO^\dagger M} \alpha$  iff  $E \models \alpha$ .*

**Proof.** From left to right we check immediately  $E$ -validity of all  $OM$ -axioms and rules ( $GOM$ ,  $GO^\dagger M$ ). For obtaining the proof of the claim from right to left we putting into the correspondence to each element  $x$  of an algebra  $E$  the hereditary set  $[x]$ . Thus, we obtain an algebra  $E^+$  which will be an orthomodular lattice. Then we define  $E^+$ -valuation as a function  $v_c : \Phi_0 \rightarrow E^+$  by means of the formula valuation  $v_c(\pi_i) = [v(\pi_i)]$ . The rest is standard. ■

Let us define now an interpretation of the system considered above in an arbitrary quantos  $\mathbf{Q}$ . The truth-value in quantos we will call an arrow of the type  $1 \rightarrow \Omega$  and the collection of all such  $\mathbf{Q}$ -arrows will be the set  $\mathbf{Q}(1, \Omega)$ .

$\mathbf{Q}$ -valuation will be a function  $V : \Phi_0 \rightarrow \mathbf{Q}(1, \Omega)$  assigning to an every propositional variable  $\pi_i$  some truth-value  $V(\pi_i) : 1 \rightarrow \Omega$ . This function apparently might be extended to the set  $\Phi$  of all formulas:

- (a)  $V(\neg\alpha) = \neg \circ V(\alpha)$ ;
- (b)  $V(\alpha \wedge \beta) = \cap \circ \langle V(\alpha), V(\beta) \rangle$ ;

Thus, we extend the valuation  $V$  in such a way that to each sentence  $\alpha$  corresponds some  $\mathbf{Q}$ -arrow  $V(\alpha) : 1 \rightarrow \Omega$ .  $\mathbf{Q}$ -validity of  $\alpha$  (which is denoted  $\mathbf{Q} \models \alpha$ ) means that  $V(\alpha) = true : 1 \rightarrow \Omega$  for all  $V$ .

Since in quantos we have  $Sub(d) \cong Hom(d, \Omega)$  then  $Sub(d) \cong \mathbf{Q}(d, \Omega)$ , i.e. bringing into correspondence with some subobject  $f$  its character  $\chi_f$  we transfer the structure of orthomodular lattice from  $Sub(d)$  on  $\mathbf{Q}(d, \Omega)$ . The connection between quantos semantics and theory considered as in case of Heyting algebra (cf. [Goldblatt, 1979]) consists in that for any quantos

$$\mathbf{Q} \models \alpha \text{ iff } \mathbf{Q}(1, \Omega) \models \alpha \text{ iff } Sub(1) \models \alpha$$

Hence, the validity in any quantos  $\mathbf{Q}$  is equal to the validity in orthomodular lattice  $\mathbf{Q}(1, \Omega)$  and  $Sub(1)$ . This implies the following theorem:

**Theorem 2.** *If  $\vdash_{OM, GOM, GO^\dagger M} \alpha$  then for any quantos  $\mathbf{Q}$  we have  $\mathbf{Q} \models \alpha$ .*

**Proof.** Let  $\alpha$  be some  $OM$ -,  $GOM$ -,  $GO^\dagger M$ -theorem. Then  $\alpha$  is valid in orthomodular lattice by the theorem 11. In particular,  $\mathbf{Q}(1, \Omega) \models \alpha$  from where  $\mathbf{Q} \models \alpha$  according to the previous claim. ■

In orthomodular lattice an introduction of implication as conditional connective is problematic by many reasons (cf. [Dalla Chiara, Giuntini, 2002, p.

146]). One of the best approximation for a material conditional in quantum logic is so-called *Sasaki arrow* (or *Sasaki hook*) which was originally proposed by P. Mittelstaedt and P.D. Finch and was further investigated by G. Hardegree. Sasaki arrow is usually introduced by the following definition:

$$\text{(SH)} \quad a \rightarrow_S b = a^\perp \vee (a \wedge b)$$

In fact, it is an algebraic counterpart of our  $a \supset_3 b$ . Sasaki arrow has the following interesting properties [Hardegree, 1981, p. 4]:

$$\text{(c1)} \quad \text{if } a \leq b \text{ then } a \rightarrow_S b = 1;$$

$$\text{(c2)} \quad a \wedge (a \rightarrow_S b) \leq b;$$

$$\text{(c3)} \quad b^\perp \wedge (a \rightarrow_S b) \leq a^\perp;$$

$$\text{(c4)} \quad a \wedge b^\perp \leq (a \rightarrow_S b)^\perp;$$

$$\text{(c5)} \quad \text{there is a binary operation } + \text{ such that for any } a, b, c \text{ we have } a + b \leq c \text{ iff } a \leq b \rightarrow_S c.$$

An operation  $+$  would be defined in orthomodular lattice by means of the following identity:

$$\text{(S)} \quad a + b = (a \vee b^\perp) \wedge b$$

In essence, (c5) corresponds the residuation  $a \wedge b \leq c$  iff  $a \leq b \rightarrow c$  in Boolean and Heyting algebras which allows to consider them categorically as Cartesian closed finitely cocomplete preorder categories. But it is easily can be seen that in orthomodular lattice one might also define exponentiation with Sasaki arrow playing the role of exponential.

**Lemma 3.** *An orthomodular lattice with Sasaki arrow categorically should be considered as Cartesian closed finitely cocomplete preorder category.*

**Proof.** One might take as exponential Sasaki arrow from the definition (SH). An evaluation arrow  $ev : (a \rightarrow_S b) \rightarrow b$  is defined according to (c2). From (c5) we obtain that for any arrow  $g : c + a \rightarrow b$  there is an arrow  $\hat{g} : c \rightarrow (a \rightarrow_S b)$  (here  $+$  is an operation from (S)). But  $c \leq a \rightarrow b$  implies  $c \wedge a \leq a \wedge (a \rightarrow b)$  (the isotone property of  $\leq$ ) hence an existence of  $\hat{g}$  implies an existence of the arrow  $c \times a \rightarrow (a \rightarrow_S b) \times a$ . By the property of  $\wedge$  we have  $c \leq (c \vee a^\perp)$  and then in virtue of the isotone property of  $\leq$  we have  $c \wedge a \leq (c \vee a^\perp) \wedge a$  but in categories this leads to an occurrence of an arrow  $c \times a \rightarrow c + a$ .

In fact, we obtain a diagram

$$\begin{array}{ccc}
 (a \rightarrow_S b) \times a & \xrightarrow{ev} & b \\
 \uparrow & & \uparrow \\
 c \times a & \longrightarrow & c + a
 \end{array}$$

In virtue of the transitivity of  $\leq$  from  $c \times a \leq c + a, c + a \leq b$  we obtain  $c \times a \leq b$  i.e. we rebuild the diagram to exponential one required. Hence, an orthomodular lattice categorically is Cartesian closed in respect to Sasaki arrow.  $\blacksquare$

G. Hardegee in [Hardegee, 1981] proposed a system of orthomodular quantum logic *OMC* where the only primitive connections are the conditional  $\supset$  (corresponding to the Sasaki hook on orthomodular lattice) and the constant 'false'  $f$ . This logic to be a smallest subset of formulas satisfying the following clauses:

$$(A1) \vdash x \supset [(x \supset y) \supset x]$$

$$(A2) \vdash [(x \supset y) \supset x] \supset x$$

$$(A3) \vdash [(x \supset y) \supset (x \supset)] \supset [(y \supset x) \supset (y \supset z)]$$

$$(A4) \vdash [(x \supset y) \supset (y \supset x)] \supset \{[(x \supset z) \supset (x \supset y)] \supset [(x \supset z) \supset (y \supset z)]\}$$

$$(A5) \vdash f \supset x$$

$$(R1) \text{ If } \vdash x, \text{ and } \vdash x \supset y, \text{ then } \vdash y.$$

$$(R2) \text{ If } \vdash x, \text{ then } \vdash y \supset x.$$

The expression ' $\vdash x$ ' is short for ' $x \in OMC$ ' which is read ' $x$  is a thesis of *OMC*'. The completeness of *OMC* is proved by yielding the Lindenbaum-Tarski algebra for *OMC* which appears to be the orthomodular lattice and the unit element of which is the equivalence class of theses of *OMC*.

In [Hardegee, 1981, p. 10] the valuation  $v$  is defined relative to  $E$  and in the role of  $E$  the Lindenbaum-Tarski algebra for *OMC* appears but since  $E^+$  is also an orthomodular lattice then it is easy to reformulate the valuation  $v$  for the case of  $E^+$ .

In this case we can enrich the Q-valuation with the point

$$(3) v(\alpha \supset \beta) = v(\alpha) \supset_3 v(\beta)$$

and extend our theorem 10 to

**Theorem 3.** *If  $\vdash_{OM, GOM, GO^+M, OMC} \alpha$  then for any quantos  $\mathbb{Q}$  we have  $\mathbb{Q} \models \alpha$ .*

An open question is whether there are systems of quantum logic with diverse conditionals corresponding  $\supset_1, \supset_2, \supset_4$  or  $\supset_5$ .

#### 4. Interpretation of Quantum Logic in quantos $[\mathbb{E}, \mathbb{QSets}]$

In order to build the category  $[\mathbb{E}, \mathbb{QSets}]$  as a quantos we consider the functor  $\underline{\Omega} : \mathbb{E} \rightarrow \mathbb{QSet}$  which will represent the classifying object in quantos  $[\mathbb{E}, \mathbb{QSets}]$ . Hereafter we will use  $\mathbb{E}$  both as an algebra and the category. For any functor  $\underline{F} : \mathbb{E} \rightarrow \mathbb{QSets}$  we denote by  $\underline{F}_p$  the value  $\underline{F}(p)$  of functor  $\underline{F}$  for object  $p$  from  $\mathbb{E}$ . For any  $q$  and  $p$  such that  $p \leq q$  a functor  $\underline{F}$  defines the function from  $\underline{F}_p$  to  $\underline{F}_q$  which we denote  $\underline{F}_{pq}$ . A functor  $\underline{F}$  will be treated as the collection  $\{\underline{F}_p : p \in \mathbb{E}\}$  of sets indexed by elements from an algebra  $\mathbb{E}$  and endowed with the transition mapping  $\underline{F}_{pq} : \underline{F}_p \rightarrow \underline{F}_q$  under  $p \leq q$  (in particular,  $\underline{F}_{pp}$  will an identity function on  $\underline{F}_p$ ).

We continue in this fashion putting  $\underline{\Omega} = [p]^+$  and for  $p$  and  $q$  such that  $p \leq q$  the function  $\underline{\Omega}_{pq} : \underline{\Omega}_p \rightarrow \underline{\Omega}_q$  maps every  $S \in [p]^+$  into  $S \cap [q] \in [q]^+$ , i.e.  $\underline{\Omega}_{pq}(S) = S_q$ .

A constant functor  $\underline{1} : \mathbb{E} \rightarrow \mathbb{QSets}$  which is a terminal object of the category  $[\mathbb{E}, \mathbb{QSets}]$  might be defined with a help of conditions  $\underline{1}_p = \{0\}$  for  $p \in \mathbb{E}$  and  $\underline{1}_{pq} = id_{\{0\}}$  under  $p \leq q$ . A subobject classifier  $true : \underline{1} \rightarrow \underline{\Omega}$  is a natural transformation whose  $p$ -th component  $true_p : \{0\} \rightarrow \underline{\Omega}_p$  will be determined by the equality  $true_p(0) = [p]$ . Thus, the function  $true$  chooses the greatest element from every orthomodular lattice of  $[p]^+$  type.

Let  $\tau : \underline{F} \xrightarrow{\bullet} \underline{G}$  be an arbitrary subobject of  $[\mathbb{E}, \mathbb{QSets}]$ -object  $\underline{G}$ . An every component  $\tau_p$  is injective and can be treated as the inclusion function  $\underline{F}_p \hookrightarrow \underline{G}_p$ . The  $p$ -th component  $(\chi_\tau)_p : \underline{G}_p \rightarrow [p]^+$  of a characteristic arrow  $\chi_\tau : \underline{G} \xrightarrow{\bullet} \underline{\Omega}$  will be defined by the equality

$$(\chi_\tau)_p(x) = \{q : p \leq q \text{ and } \underline{G}_{pq}(x) \in \underline{F}_q\}$$

for every  $x \in \underline{F}_p$ .

Now we construct truth arrows in a quantos  $[\mathbb{E}, \mathbb{QSets}]$ . Let us start with an arrow *false*.

An initial object  $\underline{0} : \mathbb{E} \rightarrow \mathbb{QSets}$  of category  $[\mathbb{E}, \mathbb{QSets}]$  is the constant functor such that  $\underline{0}_p = \emptyset$  and  $\underline{0}_{pq} = id_\emptyset$  for  $p \leq q$ . Components of a natural transformation  $\underline{0} \xrightarrow{\bullet} \underline{1}$  are the inclusions  $\emptyset \hookrightarrow \{0\}$  (the same component for any



$p$ ). According to the usual definition an arrow *false* is the characteristic arrow of subobject  $! : \underline{0} \xrightarrow{\bullet} \underline{1}$ . For its component  $false_p : \{0\} \rightarrow \underline{\Omega}_p$  we have  $false_p(0) = \{q : p \leq q \text{ and } \underline{1}_{pq}(0) \in \underline{0}_q\} = \{q : p \leq q \text{ and } \underline{0} \in \emptyset\} = \emptyset$  and hence a natural transformation chooses the null element from every orthomodular lattice.

Conjunction can be handled in same way as in case of topos  $[\mathbf{P}, \mathbf{Sets}]$  where  $\mathbf{P}$  is a Heyting algebra (cf. [Goldblatt, 1979]), i.e. we, in fact, need for  $\cap : \underline{\Omega} \times \underline{\Omega} \rightarrow \underline{\Omega}$  the definitions of their  $p$ -th components in a form of

$$\cap_p(\langle S, T \rangle) = S \cap T.$$

The negation is  $\neg : \underline{\Omega} \xrightarrow{\bullet} \underline{\Omega}$  whose  $p$ -th component  $\neg_p : \underline{\Omega}_p \rightarrow \underline{\Omega}_p$  in case of indentifying  $false_p$  with the inclusion  $\{\emptyset\} \hookrightarrow \underline{\Omega}_p$  (and since  $\neg : \underline{\Omega} \xrightarrow{\bullet} \underline{\Omega}$  is a characteristic arrow of subobject *false*) is as follows:

$$\neg_p(S) = \{q : p \leq q \text{ and } \underline{\Omega}_{pq}(S) \in \{\emptyset\}\} = \{q : p \leq q \text{ and } S \cap [q] = \{\emptyset\} = [p] \cap \neg S = (\neg S)_p\}.$$

The conditional is  $\supset_3 : \underline{\Omega} \times \underline{\Omega} \rightarrow \underline{\Omega}$  whose  $p$ -th component  $\supset_{3p} : \underline{\Omega}_p \times \underline{\Omega}_p \rightarrow \underline{\Omega}_p$  will be  $\cup_p(\neg S)_p, (S \cap T)_p$  according to the polinomiality of  $\supset_3$ .

Finally, we will call  $[\mathbf{E}, \mathbf{QSets}]$ -valuation a function  $V : \Phi_0 \rightarrow [\mathbf{E}, \mathbf{QSets}](1, \underline{\Omega})$  assigning to every propositional variable  $\pi_i$  some truth-value  $V(\pi_i) : 1 \xrightarrow{\bullet} \underline{\Omega}$ . This function apparently might be extended to the set  $\Phi$  of all formulas:

- (a)  $V(\neg\alpha) = \neg \circ V(\alpha)$ ;
- (b)  $V(\alpha \wedge \beta) = \cap \circ \langle V(\alpha), V(\beta) \rangle$ ;
- (c)  $V(\alpha \supset \beta) = \supset_3 \circ \langle V(\alpha), V(\beta) \rangle$

We say that the formula  $\alpha$  be  $[\mathbf{E}, \mathbf{QSets}]$ -valid (we write  $[\mathbf{E}, \mathbf{QSets}] \models \alpha$ ) if  $V(\alpha) = true : \underline{1} \rightarrow \underline{\Omega}$  for all  $[\mathbf{E}, \mathbf{QSets}]$ -valuations  $V$ .

Using valuation  $v : \Phi_0 \rightarrow E$  from above it is easy to prove at the same way the following theorem:

**Theorem 4.** *For any quantos  $[\mathbf{E}, \mathbf{QSets}]$ ,  $[\mathbf{E}, \mathbf{QSets}] \models \alpha$  iff  $\vdash_{OM, GOM, GO^\dagger M, OMC} \alpha$  (i.e.  $\alpha$  is provable in  $OM, GOM, GO^\dagger M, OMC$ ).*

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