# Syntax and semantics of simple paracomplete logics[1]

VLADIMIR M. POPOV, VASILIY O. SHANGIN

**ABSTRACT.** For an arbitrary fixed element $\beta$ in $\{1, 2, 3, \ldots \omega\}$ both a sequent calculus and a natural deduction calculus which axiomatise simple paracomplete logic $\mathrm{I}_{2,\beta}$ are built. Additionally, a valuation semantic which is adequate to logic $\mathrm{I}_{2,\beta}$ is constructed. For an arbitrary fixed element $\gamma$ in $\{1, 2, 3, \ldots\}$ a cortege semantic which is adequate to logic $\mathrm{I}_{2,\gamma}$ is described. A number of results obtainable with the axiomatisations and semantics in question are formulated.

*Keywords:* paracomplete logic, paraconsistent logic, cortege semantics, valuation semantics, sequent calculus, natural deduction calculus

We study logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\ldots \mathrm{I}_{2,\omega}$ presented in [8]. These logics are paracomplete counterparts of paraconsistent logics $\mathrm{I}_{1,1}$, $\mathrm{I}_{1,2}$, $\mathrm{I}_{1,3}$, $\ldots \mathrm{I}_{1,\omega}$ from [7]. In the paper, (a) simple paracomplete logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\ldots \mathrm{I}_{2,\omega}$ are defined (see [8]); these logics form (in the order indicated above) a strictly decreasing (in terms of the set-theoretic inclusion) sequence of logics, (b) for any $j$ in $\{0, 1, 2, 3, \ldots \omega\}$ both a sequent calculus $\mathrm{GI}_{2,j}$ (see [10]) and a natural deduction calculus $\mathrm{NI}_{2,j}$ which axiomatise logic $\mathrm{I}_{2,j}$ are formulated, (c) for any $j$ in $\{1, 2, 3, \ldots \omega\}$, we propose a valuation semantics for logic $\mathrm{I}_{2,j}$ (see [9]), (d) for any $j$ in $\{1, 2, 3, \ldots\}$, we propose a cortege semantics for logic $\mathrm{I}_{2,j}$ (see [9]). Below there are some results obtained with the semantics and calculi in question.

The language $L$ of each logic in the paper is a standard propositional language with the following alphabet: $\{\&, \vee,$

[1]The paper is supported by Russian Foundation for Humanities, project №10-03-00570a and project №13-03-00088a (both authors).

$\supset, \neg, (, ), p_1, p_2, p_3, \ldots\}$. As it is expected, &, $\vee$, $\supset$ are binary logical connectives in $L$, $\neg$ is a unary logical connective in $L$, brackets (, ) are technical symbols in $L$ and $p_1, p_2, p_3, \ldots$ are propositional variables in $L$. A definition of $L$-formula is as usual. Below, we say 'formula' instead of '$L$-formula' only and adopt the convention on omitting brackets as in [4]. A formula is said to be quasi-elemental iff no logical connective in $L$ other than $\neg$ occurs in it. A length of a formula $A$ is, traditionally, said to be the number of all occurrences of the logical connectives in $L$ in $A$. We denote the rule of modus ponens in $L$ by MP and the rule of substitution of a formula into a formula instead of a propositional variable in $L$ by Sub. A logic is said to be a non-empty set of formulas closed under MP and Sub. A theory for logic **L** is said to be a set of formulas including logic **L** and closed under MP. It is understood that the set of all formulas is both a logic and a theory for any logic. The set of all formulas is said to be a trivial theory. A complete theory for logic **L** is said to be a theory T for logic **L** such that, for some formula $A$, $A \in \mathrm{T}$ or $\neg A \in \mathrm{T}$. A paracomplete theory for logic **L** is said to be a theory T for logic **L** such that T is not a complete theory and any complete theory for logic **L**, which includes T, is a trivial theory. A paracomplete logic is said to be a logic **L** such that there exists a paracomplete theory for logic **L**. Simple paracomplete logic is said to be a paracomplete logic **L** such that for any paracomplete theory T for logic **L** holds true: there exists a quasi-elemental formula $A$ such that neither $A$, nor $\neg A$ belongs to T.

Let us agree that anywhere in the paper: $\alpha$ is an arbitrary element in $\{0, 1, 2, 3, \ldots \omega\}$, $\beta$ is an arbitrary element in $\{1, 2, 3, \ldots \omega\}$, $\gamma$ is an arbitrary element in $\{1, 2, 3, \ldots\}$. We define calculus $\mathrm{HI}_{2,\alpha}$. This calculus is Hilbert-type calculi, the language of $\mathrm{HI}_{2,\alpha}$ is $L$. $\mathrm{HI}_{2,\alpha}$ has MP as the only rule of inference. The notion of a derivation in $\mathrm{HI}_{2,\alpha}$ (of a proof in $\mathrm{HI}_{2,\alpha}$, in particular) is defined as usual; and for $\mathrm{HI}_{2,\alpha}$, both notion of a formula derivable from the set of formulas in this calculus and a notion of a formula provable in this calculus are defined as usual. Now we only need to define the set of axioms of $\mathrm{HI}_{2,\alpha}$.

A formula belongs to the set of axioms of calculus $\mathrm{HI}_{2,\alpha}$ iff it is one of the following forms (hereafter, $A$, $B$, $C$ denote formulas):

(I) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$, (II) $A \supset (A \vee B)$, (III) $B \supset (A \vee B)$, (IV) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$, (V) $(A\&B) \supset A$, (VI) $(A\&B) \supset B$, (VII) $(C \supset A) \supset ((C \supset B) \supset (C \supset (A\&B)))$, (VIII) $(A \supset (B \supset C)) \supset ((A\&B) \supset C)$, (IX) $((A\&B) \supset C) \supset (A \supset (B \supset C))$, (X) $((A \supset B) \supset A) \supset A$, (XI, $\alpha$) $(E \supset \neg(B \supset B)) \supset \neg E$, where $E$ is formula which is not a quasi-elemental formula of a length less than $\alpha$, (XII) $\neg A \supset (A \supset B)$.

Let us agree that, for any $j$ in $\{0, 1, 2, 3, \ldots \omega\}$, $\mathrm{I}_{2,j}$ is the set of formulas provable in $\mathrm{HI}_{2,j}$.

The following theorems 1 and 2 are shown.

THEOREM 1. *Sets $I_{2,0}$, $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\ldots I_{2,\omega}$ are logics, and, for any $k$ and $l$ in $\{0, 1, 2, 3, \ldots \omega\}$, if $k < l$, then $I_{2,l} \subseteq I_{2,k}$.*

THEOREM 2. *Logic $I_{2,0}$ is the set of the classical tautologies in $L$.*

Let us establish connections between logics $\mathrm{I}_{2,1}$, $\mathrm{I}_{2,2}$, $\mathrm{I}_{2,3}$, $\ldots \mathrm{I}_{2,\omega}$ and logic $\mathrm{I}_{2,0}$ (that is, the classical propositional logic in $L$).

Let $\varphi$ be a mapping of the set of all formulas into itself satisfying the following conditions: (1) $\varphi(p)$ is not a quasi-elemental formula, for any propositional variable $p$ in $L$, (2) for any propositional variable $p$ in $L$, formulas $p \supset \varphi(p)$ and $\varphi(p) \supset p$ belong to logic $\mathrm{I}_{2,0}$, (3) $\varphi(B \circ C) = \varphi(B) \circ \varphi(C)$, for any formulas $B$, $C$ and for any binary logical connective $\circ$ in $L$, (4) $\varphi(\neg B) = \neg\varphi(B)$, for any formula $B$.

Following these conditions, theorem 3 is shown.

THEOREM 3. *For any $j$ in $\{1, 2, 3, \ldots \omega\}$ and for any formula $A$: $A \in I_{2,0}$ iff $\varphi(A) \in I_{2,j}$.*

Let now $\psi$ be such a mapping the set of all formulas into itself satisfying the following conditions: (1) $\psi(p) = p$, for any propositional variable $p$ in $L$, (2) $\psi(B \circ C) = \psi(B) \circ \psi(C)$, for any formulas $B$, $C$ and for any binary logical connective $\circ$ in $L$, (3) $\psi(\neg B) = \psi(B) \supset \neg(p_1 \supset p_1)$, for any formula $B$.

Following these conditions, theorem 4 is shown.

THEOREM 4. *For any $j$ in $\{1, 2, 3, \ldots \omega\}$ and for any formula $A$: $A \in I_{2,0}$ iff $\psi(A) \in I_{2,j}$.*

Let us now show a method to build up a sequent calculus $\mathrm{GI}_{2,\beta}$ which axiomatises logic $\mathrm{I}_{2,\beta}$. Calculus $\mathrm{GI}_{2,\beta}$ (see [10]) is a Gentzentype sequent calculus. Sequents are of the form $\Gamma \to \Delta$ (hereafter, $\Gamma$, $\Delta$, $\Sigma$ and $\Theta$ denote finite sequences of formulas). The set of basic sequents of $GI_{2,\beta}$ is the set of all sequents of the form $A \to A$. The only rules of $GI_{2,\beta}$ are the rules R1-R15, R16($\beta$), R17 listed below.

$$\frac{\Gamma, A, B, \Delta \to \Theta}{\Gamma, B, A, \Delta \to \Theta}\ \text{R1}, \quad \frac{\Gamma \to \Delta, A, B, \Theta}{\Gamma \to \Delta, B, A, \Theta}\ \text{R2}, \quad \frac{A, A, \Gamma \to \Theta}{A, \Gamma \to \Theta}\ \text{R3},$$

$$\frac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}\ \text{R4}, \quad \frac{\Gamma \to \Theta}{A, \Gamma \to \Theta}\ \text{R5}, \quad \frac{\Gamma \to \Theta}{\Gamma \to \Theta, A}\ \text{R6},$$

$$\frac{\Gamma \to \Delta, A \qquad B, \Sigma \to \Theta}{A \supset B, \Gamma, \Sigma \to \Delta, \Theta}\ \text{R7}, \quad \frac{A, \Gamma \to \Theta, B}{\Gamma \to \Theta, A \supset B}\ \text{R8},$$

$$\frac{A, \Gamma \to \Theta}{A\&B, \Gamma \to \Theta}\ \text{R9}, \quad \frac{A, \Gamma \to \Theta}{B\&A, \Gamma \to \Theta}\ \text{R10}, \quad \frac{\Gamma \to \Theta, A \qquad \Gamma \to \Theta, B}{\Gamma \to \Theta, A\&B}\ \text{R11},$$

$$\frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B}\ \text{R12}, \quad \frac{\Gamma \to \Theta, A}{\Gamma \to \Theta, B \vee A}\ \text{R13}, \quad \frac{A, \Gamma \to \Theta \qquad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}\ \text{R14},$$

$$\frac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}\ \text{R15},$$

$$\frac{E, \Gamma \to \Theta}{\Gamma \to \Theta, \neg E}\ \text{R16}(\beta),$$ where $E$ is a formula which is not a quasi-elemental formula of a length less than $\beta$,

$$\frac{\Gamma \to \Delta, A \qquad A, \Sigma \to \Theta}{\Gamma, \Sigma \to \Delta, \Theta}\ \text{R17 (cut rule)}$$

A derivation in calculus $GI_{2,\beta}$ is defined in a standard sequent calculus fashion. The definition of a sequent provable in $GI_{2,\beta}$ is as usual. The cut-elimination theorem is shown (by Gentzen's method presented in [3]) to be valid in $GI_{2,\beta}$.

The following theorem 5 is shown.

THEOREM 5. *For any $j$ in $\{1, 2, 3, \dots \omega\}$ and for any formula $A$: $A \in I_{2,j}$ iff a sequent $\to A$ is provable in $GI_{2,j}$.*

Let us now show a method to build up a Fitch-style natural deduction calculus $NI_{2,\beta}$ which axiomatises logic $I_{2,\beta}$.

The set of $NI_{2,\beta}$-rules is as follows, where $[A]C$ denotes a derivation of a formula $C$ from a formula $A$.

$$\frac{C\&C_1}{C}\ \&_{el1} \qquad \frac{C\&C_1}{C_1}\ \&_{el2} \qquad \frac{C, C_1}{C\&C_1}\ \&_{in}$$

$$\frac{C \vee C_1,\ [C]C_2\ [C_1]C_2}{C_2} \vee_{el} \qquad \frac{C}{C \vee C_1} \vee_{in1} \qquad \frac{C_1}{C \vee C_1} \vee_{in2}$$

$$\frac{C \supset C_1,\ C}{C_1} \supset_{el} \qquad \frac{[C]C_1}{C \supset C_1} \supset_{in} \qquad \frac{[A \supset B]A}{A} \supset_p$$

$$\frac{[E]\ \neg(C \supset C)}{\neg E} \neg_{in1(\beta)},$$ where $E$ is a formula which is not a quasi-elemental formula of a length less than $\beta$.

$$\frac{\neg C_1,\ C_1}{C} \neg_{in2}$$

A derivation in $\mathrm{NI}_{2,\beta}$ is defined in a standard natural deduction calculus fashion.

The following theorem 6 is shown.

THEOREM 6. *For any $j$ in $\{1, 2, 3, \ldots \omega\}$ and for any formula $A$ : $A \in I_{2,j}$ iff $A$ is provable in $NI_{2,j}$.*

The proof search procedures which were proposed to the classical and a variety of non-classical logics are applicable [1, 2].

Let us construct $\mathrm{I}_{2,\beta}$-valuation semantics for $\mathrm{I}_{2,\beta}$. By $\mathrm{Q}_\beta$ we denote the set of all quasi-elemental formulas of a length less or equal to $\beta$. By $\mathrm{I}_{2,\beta}$-valuation we mean any mapping $v$ set $\mathrm{Q}_\beta$ into the set $\{0,\ 1\}$ such that, for any quasi-elemental formula $e$ of a length less than $\beta$, if $v(e) = 1$, then $v(\neg e) = 0$. Let Form denote the set of all formulas and let $\mathrm{Val}_{2,\beta}$ denote the set of all $\mathrm{I}_{2,\beta}$-valuations. It can be shown there exists a unique mapping (denoted by $\xi_{2,\beta}$ ) satisfying the following six conditions: (1) $\xi_{2,\beta}$ is a mapping a Cartesian product Form $\times$ $\mathrm{Val}_{2,\beta}$ into the set $\{1,\ 0\}$, (2) for any quasi-elemental formula $Y$ in $\mathrm{Q}_\beta$ and any $\mathrm{I}_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(Y, v) = v(Y)$, (3) for any formulas $A$, $B$ and any $\mathrm{I}_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A\&B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ and $\xi_{2,\beta}(B) = 1$, (4) for any formulas $A$, $B$ and any $\mathrm{I}_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A \vee B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ or $\xi_{2,\beta}(B) = 1$, (5) for any formulas $A$, $B$ and any $\mathrm{I}_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A \supset B, v) = 1$ iff $\xi_{2,\beta}(A) = 0$ or $\xi_{2,\beta}(B) = 1$, (6) for any formula $A$ which is not a quasi-elemental formula of a length less than $\beta$, and for any $\mathrm{I}_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(\neg A, v) = 1$ iff $\xi_{2,\beta}(A, v) = 0$. A formula $A$ is said to be $\mathrm{I}_{2,\beta}$-valid iff for any $\mathrm{I}_{2,\beta}$-valuation $v$, $\xi_{2,\beta}(A, v) = 1$.

The following theorems 7 and 8 are shown.

THEOREM 7. *For any $j$ in $\{1, 2, 3, \ldots \omega\}$, for any formula $A$, for any set $\Gamma$ of formulas: formula $A$ is derivable from $\Gamma$ in $HI_{2,j}$ iff for* *any $I_{2,j}$-valuation $v$, if for any formula $B$ in $\Gamma$, $\xi_{2,j}(B,v)=1$, then $\xi_{2,j}(A,v)=1$.*

THEOREM 8. *For any $j$ in $\{1,2,3,\ldots\omega\}$ and for any formula $A$, $A\in I_{2,j}$ iff formula $A$ is $I_{2,j}$-valid.*

It should be noted that the proposed $\mathrm{I}_{2,\beta}$-valuation semantics is consistent to the requirements, which, in our point of view, N.A. Vasiliev considers to be necessary in [11]: (1) no proposition cannot be true and false at once, (2) in general case, a value of the proposition that is a negation of a proposition $P$, is not determined by the value of $P$.

Let us construct $\mathrm{I}_{2,\gamma}$-cortege semantics for $\mathrm{I}_{2,\gamma}$. By $\mathrm{I}_{2,\gamma}$-cortege we mean an ordered $\gamma+1$-tuplet of elements of the set $\{1,\ 0\}$ such that for any two neighboring members of this ordered $\gamma+1$-tuplet, at least one of them is 0. By a designated $\mathrm{I}_{2,\gamma}$-cortege we mean $\mathrm{I}_{2,\gamma}$-cortege, where the first member is 1. By $\mathrm{S}_{2,\gamma}$ we denote the set of all $\mathrm{I}_{2,\gamma}$-corteges and by $\mathrm{D}_{2,\gamma}$ we denote the set of all designated $\mathrm{I}_{2,\gamma}$-corteges. By a normal $\mathrm{I}_{2,\gamma}$-cortege we mean $\mathrm{I}_{2,\gamma}$-cortege such that any two neighboring members of this $\mathrm{I}_{2,\gamma}$-cortege are different. By a single $\mathrm{I}_{2,\gamma}$-cortege we mean a normal $\mathrm{I}_{2,\gamma}$-cortege such that the first member of it is 1. By a zero $\mathrm{I}_{2,\gamma}$-cortege we mean a normal $\mathrm{I}_{2,\gamma}$-cortege such that the first member of it is 0.

It is clear that there exists a unique single $\mathrm{I}_{2,\gamma}$-cortege (denoted by $\mathbf{1}_\gamma$) and there exists a unique zero $\mathrm{I}_{2,\gamma}$-cortege (denoted by $\mathbf{0}_\gamma$). It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\&_{2,\gamma}$) satisfying the following condition, for any $X$, $Y$ in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege $X$ is 1 and the first member of $\mathrm{I}_{2,\gamma}$-cortege $Y$ is 1 then $X\&_{2,\gamma}Y$ is $\mathbf{1}_\gamma$; otherwise, $X\&_{2,\gamma}Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\vee_{2,\gamma}$) satisfying the following condition, for any $X$ and $Y$ in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege $X$ is 1 or the first member of $\mathrm{I}_{2,\gamma}$-cortege $Y$ is 1 then $X\vee_{2,\gamma}Y$ is $\mathbf{1}_\gamma$; otherwise, $X\vee_{2,\gamma}Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique binary operation on $\mathrm{S}_{2,\gamma}$ (denoted by $\supset_{2,\gamma}$) satisfying the following condition, for any $X$ and $Y$ in $\mathrm{S}_{2,\gamma}$: if the first member of $\mathrm{I}_{2,\gamma}$-cortege $X$ is 0 or the first member of $\mathrm{I}_{2,\gamma}$-cortege $Y$ is 1 then $X\supset_{2,\gamma}Y$ is $\mathbf{1}_\gamma$; otherwise, $X\supset_{2,\gamma}Y$ is $\mathbf{0}_\gamma$. It can be shown that there exists a unique unary operation on $S_{2,\gamma}$ (denoted by $\neg_{2,\gamma}$) satisfying the following condition, for any $I_{2,\gamma}$-cortege $< x_1, x_2, \dots, x_\gamma, x_{\gamma+1} >$: if $x_{\gamma+1}$ is 1 then $\neg_{2,\gamma}(< x_1, x_2, \dots, x_\gamma, x_{\gamma+1} >) =< x_2, \dots, x_\gamma, x_{\gamma+1}, 0 >$ and if, if $x_{\gamma+1}$ is 0, then $\neg_{2,\gamma}(< x_1, x_2, \dots, x_\gamma, x_{\gamma+1} >) = < x_2, \dots, x_\gamma, x_{\gamma+1}, 1 >$.

It is clear that $< S_{2,\gamma}, D_{2,\gamma}, \&_{2,\gamma}, \vee_{2,\gamma}, \supset_{2,\gamma}, \neg_{2,\gamma} >$ is a logical matrix. This logical matrix (denoted by $M_{2,\gamma}$) is said to be $I_{2,\gamma}$-matrix. $M_{2,\gamma}$-valuation is said to be a mapping the set of all propositional variables in $L$ into $S_{2,\gamma}$. The set of all $M_{2,\gamma}$-valuations is denoted by $ValM_{2,\gamma}$. It can be shown that there exists a unique mapping (denoted by $\xi M_{2,\gamma}$) satisfying the following conditions: (1) $\xi M_{2,\gamma}$ is a mapping a Cartesian product Form x $ValM_{2,\gamma}$ into the set $S_{2,\gamma}$, (2) for any propositional variable $p$ in $L$ and for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(p, w) = w(p)$, (3) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(A\&B, w) = \xi M_{2,\gamma}(A, w)\&_{2,\gamma}\xi M_{2,\gamma}(B, w)$, (4) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(A \vee B, w) = \xi M_{2,\gamma}(A, w) \vee_{2,\gamma} \xi M_{2,\gamma}(B, w)$, (5) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(A \supset B, w) = \xi M_{2,\gamma}(A, w) \supset_{2,\gamma} \xi M_{2,\gamma}(B, w)$, (6) for any formula $A$ and for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(\neg A, w) = \neg_{2,\gamma}\xi M_{2,\gamma}(A, w)$.

A formula $A$ is said to be $M_{2,\gamma}$-valid iff for any $M_{2,\gamma}$-valuation $w$, $\xi M_{2,\gamma}(A, w) \in D_{2,\gamma}$.

The following theorems 9–11 are shown.

THEOREM 9. *For any $j$ in $\{1, 2, 3, \dots\}$, for any formula $A$ and for any set $\Gamma$ of formulas, formula $A$ is derivable from $\Gamma$ in $HI_{2,j}$ iff for any $M_{2,j}$-valuation $w$, if for any formula $B$ from $\Gamma$, $\xi M_{2,j}(B, w) \in D_{1,j}$ then $\xi M_{2,j}(A, w) \in D_{2,j}$.*

THEOREM 10. *For any $j$ in $\{1, 2, 3, \dots\}$ and for any formula $A$, $A \in I_{2,j}$ iff $A$ is $M_{2,j}$-valid.*

THEOREM 11. *For any $j$ in $\{1, 2, 3, \dots\}$ and for any formula $A$, $A$ is $M_{2,j}$-valid iff for any $M_{2,j}$-valuation $w$, $\xi M_{1,j}(A, w) \in \mathbf{1}_j$.*

The following theorems 12–19 are shown with the help of the axiomatisations and semantics presented in the paper.

THEOREM 12. *Logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\dots I_{2,\omega}$ are simple paracomplete logics.*

THEOREM 13. *For any $j$ and $k$ in $\{1, 2, 3, \dots \omega\}$, if $j \neq k$ then $I_{2,j} \neq I_{2,k}$.*

THEOREM 14. *For any $j$ in $\{1, 2, 3, \dots \omega\}$, the positive fragment of logic $I_{2,j}$ is equal to the positive fragment of logic $I_{2,0}$.*

THEOREM 15. *For any $j$ in $\{1, 2, 3, \dots \omega\}$, logic $I_{2,j}$ is decidable.*

THEOREM 16. *For any $j$ in $\{1, 2, 3, \dots\}$, logic $I_{2,j}$ is finitely-valued.*

THEOREM 17. *Logic $I_{2,\omega}$ is not finitely-valued.*

THEOREM 18. *Logic $I_{2,\omega}$ is equal to the intersection of logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\dots$*

THEOREM 19. *There is a continuum of logics which include $I_{2,\omega}$ and are included in $I_{2,1}$.*

## References


[1] BOLOTOV, A., GRIGORYEV, O., and SHANGIN, V., Automated Natural Deduction for Propositional Linear-time Temporal Logic, *Proceedings of the 14th International Symposium on Temporal Representation and Reasoning (Time2007), Alicante, Spain, June 28-June 30*, pp.47–58, 2007.

[2] BOLOTOV, A .E., SHANGIN, V., Natural Deduction System in Paraconsistent Setting: Proof Search for PCont, *Journal of Intelligent Systems*, 21(1):1–24, 2012.

[3] GENTZEN, G., Investigations into logical deductions, *Mathematical theory of logical deduction*, Nauka Publishers, M., 1967, pp. 9–74 (in Russian).

[4] KLEENE, S. C., *Introduction to Metamathematics*, Ishi Press International, 1952.

[5] POPOV, V. M., On the logic related to A. Arruda's system V1, *Logic and Logical Philosophy*, 7:87–90, 1999.

[6] POPOV, V. M., Intervals of simple paralogics, *Proceedings of the V conference 'Smirnov Readings in Logic', June, 20-22, 2007*, M., 2007, pp. 35–37 (in Russian).

[7] POPOV, V. M., Two sequences of simple paraconsistent logics, *Logical investigations*, *Vol.* 14, M., 2007, pp. 257–261 (in Russian).

[8] Popov, V. M., Two sequence of simple paracomplete logics, *Logic today: theory, history and applications. The proceedings of X Russian conference, June, 26-28, 2008*, St.-Petersburg, SPbU Publishers, 2008, pp. 304–306 (in Russian).

[9] Popov, V. M., Semantical characterization of paracomplete logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, ..., *Logic, methodology: actual problems and perspectives. The proceedings of conference*, Rostov-on-Don, UFU Publishers, 2010, pp. 114–116 (in Russian).

[10] Popov, V. M., Sequential characterization of simple paralogics, *Logical investigations*, 16:205–220, 2010 (in Russian).

[11] Vasiliev, N. A., Imaginary (non-Aristotelian) logic, *Vasiliev N.A. Imaginary logic. Selected works*, M., Nauka Publishers, 1989, pp. 53–94 (in Russian).