Von Wright's truth-logic and around

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Devoted to Georg Henrik von Wright

abstract. In this paper von Wright's truth-logic $\mathbf{T}''$ is considered. It seems that it is a De Morgan four-valued logic **DM4** (or Belnap's four-valued logic) with endomorphism e_2. In connection with this many other issues are discussed: *twin* truth operators, a truth-logic with endomorphism g (or logic **Tr**), the lattice of extensions of **DM4**, modal logic **V2**, Craig interpolation property, von Wright–Segerberg's tense logic **W**, and so on.

Keywords: Wright's truth-logic, De Morgan four-valued logic, twin truth operators, tetravalent modal logic TML, truth logic Tr, modal logic V2, von Wright–Segerberg's tense logic

1 Four-valued classical logic $\mathbf{C}_4$ and four-valued De Morgan logic DM4

Let $\mathfrak{M}_4^C$ be a four-valued logical matrix

$$\mathfrak{M}_4^C = < \{1, b, n, 0\}, \supset, \vee, \wedge, \neg, \{1\} >$$

which is obtained from the direct product of the matrix $\mathfrak{M}$ (for classical propositional logic $\mathbf{C}_2$) with itself, i.e. $\mathfrak{M}_4^C = \mathfrak{M}_2^C \times \mathfrak{M}_2^C$, where matrix operations $\supset, \vee, \wedge, \neg$ are the following:

x	$\neg x$
1	0
b	n
n	b
0	1

$\supset$	1	b	n	0
1	1	b	n	0
b	1	1	n	n
n	1	b	1	b
0	1	1	1	1

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

Note that the set of truth-values $\{1, b, n, 0\}$ is partially-ordered in the form $0 < n,\ b < 1$, i.e. n and b are incomparable. As usual

$$x \vee y =: \neg x \supset y,$$

$$x \wedge y =: \neg(\neg x \vee \neg y),$$

$$x \equiv y =: (x \supset y) \wedge (y \supset x).$$

It is well known that matrix $\mathfrak{M}_4^C$ is characteristic for calculus $\mathbf{C}_2$. The logic with the above operations is denoted as $\mathbf{C}_4$. As usual, we will denote connectives and the similar operations by the same symbols.

Then the logic with the operations $\vee, \wedge$ and $\sim$ is called four-valued De Morgan logic **DM4**, where $\sim$ is De Morgan negation: $\sim 1 = 0, \sim b = b, \sim n = n, \sim 0 = 1$ (see [5], [9]). In another terminology, **DM4** is *Belnap's four-valued logic* [3].

2 Endomorphismus in the distributive lattices

In [6] the authors point out the fact that the modal and tense operations in a number of modal and tense logics and in corresponding algebras are expressed in terms of endomorphismus in the distributive lattices.

Let us consider one-place operations g, e_1 and e_2

x	$g(x)$	$e_1(x)$	$e_2(x)$
1	1	1	1
b	n	0	1
n	b	1	0
0	0	0	0

which are the endomorphismus in the distributive lattices:

$$f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y),$$

$$f(\neg x) = \neg f(x), f(1) = 1, f(0) = 0, f(x^{\delta}) = (f(x))^{\delta},$$ [1]

where f can be any operations from g, e_1 and e_2.

3 Von Wright's truth-logic T″

Now in the new terms introduced above we can define Wright's truth-logic. The expansion of **DM4** by the endomorphism e_2 leads to the logic which G.H. von Wright in 1985 denoted as **T″LM** and called a '*truth-logic*' (see [28]). For the sake of brevity, we will denote it as **T″**. Here a truth-operator T is the endomorphism e_2. Note that the following important definitions hold:

$$(*)\ e_1(x) =: \sim (e_2(\sim x)) \text{ and } e_2(x) =: \sim (e_1(\sim x)).$$ [2]

It is easy to show that all four-valued $J_i(x)$-operations are definable in **T″LM**, where

$$J_i(x) = \begin{cases} 1, & \text{if } x = i \\ 0, & \text{if } x \neq i \end{cases} (i = 1, n, b, 0).$$

Thus, we have:

x	$J_1(x)$	$J_b(x)$	$J_n(x)$	$J_0(x)$
1	1	0	0	0
b	0	1	0	0
n	0	0	1	0
0	0	0	0	1

[1]

$$x^{\delta} = \begin{cases} x, \text{if } \delta = 1 \\ \neg x, \text{if } \delta = 0. \end{cases}$$

[2] In [19] a four-valued 'logic of falsehood' **FL4** is formalized. In our terms it is the expansion of the language of **DM4** by the endomorphism e_1. So, in virtue of (*) logics **FL4** and **T″** are functionally equivalent.

One may easily verify that

$J_1 =: e_1(x) \wedge e_2(x),$

$J_b =: \sim e_1(x) \wedge e_2(x),$

$J_n =: e_1(x) \wedge \sim e_2(x),$

$J_0 =: \sim e_1(x) \wedge \sim e_2(x).$

Note that $e_2(x) =: J_1 \vee J_b$. Then Wright's logic $\mathbf{T}''$ is De Morgan logic **DM4** with all $J_i(x)$-operators (but, it is important, without classical negation $\neg$). Note also that in many finite modal logics the operator J_1 is the modal operator of necessity $\Box$. Then the well-known tetravalent modal logic **TML** is **DM4** with the operator $\Box$ added to its language (see especially [9][3]). So $\mathbf{T}''$ is an extension of **TML**.

Now we need some additional definitions. A finite-valued logic $\mathbf{L}_n$ with all $J_i(x)$-operators is called *truth-complete* logic, and a logic $\mathbf{L}_n$ is said to be **C**-*extending* iff in $\mathbf{L}_n$ one can functionally express the binary operations $\supset, \vee, \wedge$, and the unary negation operation, whose restrictions to the subset $\{0, 1\}$ coincide with the classical logical operations of implication, disjunction, conjunction, and negation. In virtue of result of [2] every truth-complete and **C**-extending logic has Hilbert-style axiomatization extending the $\mathbf{C}_2$. It means that Wright's $\mathbf{T}''$ logic has such an axiomatization. Moreover, it follows from [1] that we have adequate first-order axiomatization for logic $\mathbf{T}''$ with quantifiers.

It is very interesting to generalize given four-valued von Wright's logic, i.e. to consider an arbitrary finite-valued De Morgan logic with all $J_i(x)$-operators. As a result, we obtain an entirely new class of many-valued logics which I suggest to call '*Wright's many-valued logics*' and a new class algebras which I suggest to call '*Wright's algebras*'. Then again it follows from [1] that for such logics we have adequate first-order axiomatization.

[3] However, see also [5].

4 Properties of a truth-operator T and the *twin* truth operators

The following two properties of a truth-operator T are useful:

(I) $T(\sim x) \equiv \sim T(x)$

(II) $T(x) \vee T(\sim x)$ — the law of excluded middle.

Note that these two conditions are required in the Tarski's axiomatic theory of truth with a predicate symbol *True* (see [12]).

None of these conditions is fulfilled in the logic $\mathbf{T}''$. However it is interesting to consider the operations e_1 and e_2 as the twin truth operators T_1 and T_2 bearing in mind $(*)$. Then

(I') $T_1(\sim x) \equiv \sim T_2(x)$

(II') $T_1(x) \vee T_2(\sim x)$ — the law of excluded middle.

Here we must note that the main goal pursued by von Wright has been the construction of paraconsistent logic. So the choice of the operations $\sim$ and T_2 is such that the law of contradiction

$$\sim (T_2(x) \wedge T_2(\sim x))$$

is not valid in $\mathbf{T}''$. But it is interesting that this law is valid in the form

$$\sim (T_1(x) \wedge T_2(\sim x)) \quad \text{or} \quad \sim (T_2(x) \wedge T_1(\sim x)).$$

We want to stress that *von Wright's truth logic* with the twin truth operators T_1 and T_2 seems to us very interesting.

5 Logic Tr

Let us consider the expansion of **DM4** by the endomorphism g. Now the conditions (I)–(II) are fulfilled. Note that operators $\sim$ and g commute among themselves, i.e.

$$\sim g(x) \equiv \; g \sim (x).$$

Moreover, this allows to define the classical negation $\neg$:

$$\neg(x) =: \sim g(x).$$

We denote a *truth* logic with the set of operations $\{\vee, \wedge, \sim, g\}$ by **Tr**.

There is a very simple and nice axiomatization of this logic (see justification below), where the operation T is g:

(A0) Axioms of classical propositional logic $\mathbf{C}_2$.

(A1) $T(A \supset B) \equiv (TA \supset TB)$.

(A2) $\neg TA \equiv T\neg A$.

(A3) $TTA \equiv A$.

The single rule of inference: *modus ponens*.[4]

It is worth to mention that there is a generalized truth-value space in kind of *bilattice* (see [11]). Indeed, smallest nontrivial bilattice is just the four-valued Belnap's logic. In [8] M. Fitting extends a first-order language by notation for elementary arithmetic, and builds the theory of truth based on bilattice. This four-valued theory of truth is an alternative to Tarsky's approach.

Also in one case, Fitting extends this language by the operation '*conflation*' (endomorphism g).

6 Interrelations between $\mathbf{T}''$ and Tr

Let P_4 be Post's four-valued functionally complete logic (see [20]). The set operation R is called functionally *precomplete* in P_4 if every enlargement $\{R, f\} = R \cup \{f\}$ of the set R by an operation f such that $f \notin R$ and $f \in P_4$ is functionally complete.

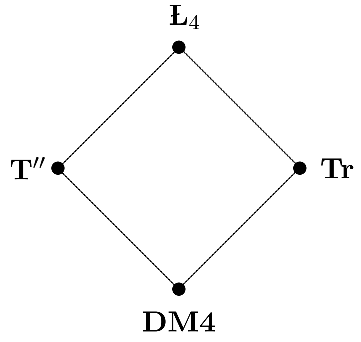
It is not difficult to prove, that the logic with the set of the operations $\{\vee, \wedge, \sim, e_2, g\}$ is four-valued Łukasiewicz logic $\mathbf{Ł}_4$ which first appeared in [15]). According to Finn's result $\mathbf{Ł}_4$ is precomplete in P_4 (see [4]). Note that in $\mathbf{Ł}_4$

[4] At the time of my report G. Sandu had asked about the logic **Tr** with the axiom $(A4)$ $TA \equiv A$. Let's denote this logic by $\mathbf{Tr^c}$. If we take the operation T as identity operation of $\mathbf{C}_2$ then the logic $\mathbf{Tr^c}$ is a conservative extension of $\mathbf{C}_2$.

$$x \vee y = max(x, y) \text{ and } x \wedge y = min(x, y),$$

i.e. the truth-values in $\mathbf{Ł}_4$ are linearly-ordered[5].

As a result, we have the following lattice of extensions of **DM4**:



7 Modal logic V2

In [25] Sobochiński presents the formula (β_2):

$$\Box p \vee \Box(p \supset q) \vee \Box(p \supset \neg q).$$

He establishes that it is not provable in **S5**, and **S5** plus (β_2) is not classical calculus $\mathbf{C}_2$. In [26] this logic is denoted by **V2**. As a consequence of Scroggs' result about *pretabularity* of **S5**[6] logic **V2** is finite-valued one. It was remarked that four-valued matrix of 'group III' from [14], i.e. matrix

$$< \{1, b, n, 0\}, \supset, \neg, \Box, \{1\} >,$$

is characteristic for **V2** (see e.g. [5, p. 190]).

In [6] it has been shown that logics **Tr** and **V2** are functionally equivalent:

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[5]In details about different finite-valued logics see in [13, ch. 5].

[6]A logic L is said to be *pretabular* if it is not finite (tabular), but its proper extension is finite. Scroggs [22] has shown that **S5** has no finite characteristic matrix but every proper normal extension does.

$\Box p =: p \wedge g(p)$,

$\Diamond p =: \neg\Box\neg p$,

$g(p) =: \Box p \vee (\neg p \wedge \Diamond p)$.[7]

Note that in [5] an algebraic semantics (named to MB-algebras) has been developed for logic **Tr (V2)**. MB-algebra is an expansion of De Morgan algebra by Boolean negation $\neg$. In this case $g(x) = \sim \neg(x) = \neg \sim (x)$. It is interesting that Pynko [21] introduces a similar algebraic structure called *De Morgan boolean algebra*. He also suggests Gentzen-style axiomatization of four-valued logic denoted by **DMB4**.

In [17] Maksimova considers all normal extensions of modal logic **S4** with the *Craig interpolation property*. From this it follows that modal logic **V2** is the *single* normal extension of modal logic **S5** with the Craig interpolation property (between **S5** and $\mathbf{C}_2$). Since the logics **Tr** and **V2** are functionally equivalent then the following theorem can be proved:

THEOREM 1. *A logic* ***Tr*** *has the Craig interpolation property.*

8 Von Wright–Segerberg's tense logic W

It is interesting that we can come to the logic **Tr** on the basis of an entirely different considerations. In [27] von Wright presents a tense logic 'And next' which deals with discrete time. In [23] Segerberg reformulates it under the name **W** and provides other proofs of completeness theorem, and decision procedure.[8]

A logic **W** is a very simple propositional logic in which a new unary operation S with the intuitive meaning of 'tomorrow' is added to the language of the classical propositional calculus. **W** is axiomatized in the following way:

(A0) Axioms of classical propositional logic $\mathbf{C}_2$.

(A1) $S(A \supset B) \equiv (SA \supset SB)$.

(A2) $\neg SA \equiv S\neg A$.

[7] However, see [23, p. 49].

[8] For detailed overview of von Wright's tense logic see Segerberg's paper [24].

The rules of inference:

R1. *Modus ponens*,

R2. *From A follows SA*.

Segerberg suggests the following Kripke-style semantics for **W** (this semantics in a simplified way is presented in [7, p. 288]). Let $N = 0, 1, 2, \ldots$ be the set of possible worlds. Valuation $v(p_i, w) = 1, 0$ ('truth', 'falsehood') for propositional variables p_i and $w \in N$. For $\supset$ and $\neg$ as usual, and for $SA : v(SA, w) = v(A, w + 1)$. Pay attention that **W** is the logic that defines the set formulas valid in N.

Concerning the logic **W** there are the following meta-logical results:

1) There is no finite axiomatization of **W** with modus ponens as sole inference rule [23].

2) Logic **W** is pretabular [7].

It is worth emphasizing that in [6] Mučhnik has devised algebraic semantics for **W**, named Bg-algebras, and has proved Stone's representation theorem for them. Here it is noted that Bg-algebra with involution, where $gg(x) = x$, corresponds to the logic **V2**. Thus we again have come to the logic **Tr**.

Note than in [18] Kripke frame, consisting two possible worlds, is represented for **V2**. Here we describe Kripke frame $\imath =< T, R >$ for **W** and **Tr**, where T is the set of instants of time.

A Kripke frame $\imath =< T, R >$ is a frame for **W** if the following conditions fulfill:

1. $\forall w \in T\ \exists v \in T wRv$

   '*from every point (instant) something is accessible*'.

2. $\forall w \in T\ \forall v_1 \in\ T\ \forall v_2 \in T(wRv_1 \,\&\, wRv_2 \Rightarrow v_1 = v_2)$

   '*from every point no more than one point is accessible*'.

And for **Tr** it is necessary to add:

3. $\forall w_1 \in T\ \forall w_2 \in T\ \forall w_3 \in T (w_1 R w_2 \,\&\, w_2 R w_3 \Rightarrow w_3 = w_1)$

   '*from every point in two steps we once again find ourselves in the same point*'.

THEOREM 2. *Logic* **W** + *axiom* (A3) $SSA \equiv A$ *and logic* **Tr** *are the same as the sets of derivable formulas.*

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