Generalization of Kalmar's method for quasi-matrix logic[1]

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abstract. Quasi-matrix logic is based on the generalization of the principles of classical logic: *bivalency* (a proposition take values from the domain $\{t \text{ (truth)}, f \text{ (falsity)}\}$); *consistency* (a proposition can not take on both values); *excluded middle* (a proposition necessarily takes some of these values); *identity* (in a complex proposition, a system of propositions, an argument the same proposition takes the same value from domain $\{t, f\}$); *matrix principle* — logical connectives are defined by matrices. As a result of our generalization, we obtain quasi-matrix logic principles: the principle of *four-valency* (a proposition takes values from domain $\{t^n, t^c, f^c, f^i\}$) or *three-valency* (a proposition takes values from domain $\{n, c, i\}$); *consistency*: a proposition can not take more than one value from $\{t^n, t^c, f^c, f^i\}$ or from $\{n, c, i\}$; the principle of *excluded fifth or fourth*; *identity* (in a complex proposition, a system of propositions, an argument the same proposition takes the same value from domain $\{t^n, t^c, f^c, f^i\}$ or domain $\{n, c, i\}$); the *quasi-matrix principle* (logical terms are interpreted as quasi-functions). Quasi-matrix logic is a logic of factual modalities.

Keywords: quasi-matrix logic, semantic completeness, decision problem, Kalmar's method

1 Kalmar's method

Well-known proof method for methateorem of semantic completeness of classical propositional calculus, which may be also treated as an approach to the solution of the decision problem, implies the proof of the following lemma:

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LEMMA 1. Assuming that D is a formula, $a_1, \ldots, a_n$ are all different variables, occurring in D, $b_1, \ldots, b_n$ are truth-values of these variables; let A_i be $a_i, \neg a_i$, depending on whether b_i takes value t or f; let D' be D or $\neg D$ depending on whether D takes value t or f with truth-values $b_1, \ldots, b_n$ variables $a_1, \ldots, a_n$. Then $A_1, \ldots, A_n \Rightarrow D'$.

($\Rightarrow$ is here a sign (symbol) for logical entailment, $\neg$ — for negation, t и f — truth and falsity, respectively.)

2 Generalization of Kalmar's method for many-valued matrix logic

At the end of the sixties of the 20-th century I was able to generalize this method for functionally complete many-valued matrix logics. (Probably the generalization of this kind had been done earlier by somebody else, but I have not heard of it up to now.)

Let's illustrate the basic principles underlying the generalization with one of the system of modal logic Sb^- constructed by me.

Logical terms of language: $\neg$, $\supset$, $\Box$, $\Diamond$. ('$\supset$', '$\Box$', '$\Diamond$' — are respectively signs for implication, necessity and possibility)

2.1 Semantics Definitions of logical terms

$\supset$	t^n	t^c	f^i	f^c
t^n	t^n	t^c	f^i	f^c
t^c	t^n	t^c	f^c	f^c
f^i	t^n	t^n	t^n	t^n
f^c	t^n	t^c	t^c	t^c

A	$\neg A$	$\Box A$	$\Diamond A$
t^n	f^i	t^n	t^n
t^c	f^c	f^c	t^c
f^i	t^n	f^i	f^i
f^c	t^c	f^c	t^c

t^n, t^c, f^c, f^i — are respectively truth-values 'necessary truth', 'contingent truth', 'contingent falsity', 'necessary falsity'. Designated values are t^n and t^c.

2.2 Formalisation

The calculus includes schemes of axioms of classical propositional calculus, **modus ponens** rule of inference and also following schemes of axioms:

$\Box A \supset A$; $\neg\Box\neg A \supset \Diamond A$; $\Diamond A \supset \neg\Box\neg A$; $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$; $\Diamond B \supset \Diamond(A \supset B)$; $\Diamond\neg A \supset \Diamond(A \supset B)$; $\Diamond(A \supset B) \supset$

$(\Box A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

For the proof of meta-theorem of semantic completeness of calculi Sb^- the following **lemma** is needed.

LEMMA 2. Assuming that D is a formula, $a_1, ..., a_n$ are all different variables occurring in D, $b_1, ..., b_n$ are truth-values of these variables. Let A_i be $\Box a_i$, $a_i \& \Diamond\neg a_i$, $\neg a_i \& \Diamond a_i$, $\neg\Diamond a_i$ depending on whether b_i is t^n, t^c, f^c or f^i. Let D' be $\Box D$, $D \& \Diamond\neg D$, $\neg D \& \Diamond D$ or $\neg\Diamond D$ depending on whether D takes value t^n, t^c, f^c or f^i with truth-values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$. Then $A_1, ..., A_n \Rightarrow D'$. (& — is here a sign for conjunction)

Lemma is proved by the use of recurrent mathematical induction.

If formula D takes designated value with all possible truth-values of its variables, then D' is $\Box D$ or $D \& \Diamond\neg D$. In each case $A_1, ..., A_n \Rightarrow D$.

Let us substitute assumption $\Box a_{i+1}$ with number $i+1$ from the set of assumptions $A_1, ..., A_n$ for the set of formulas $a_{i+1}, \neg\Diamond\neg a_{i+1}$, assumption $a_{i+1} \& \Diamond\neg a_{i+1}$ for the set of formulas $\Diamond\neg a_{i+1}, a_{i+1}$, assumption $\neg a_{i+1} \& \Diamond a_{i+1}$ for the set of formulas $\neg a_{i+1}, \Diamond a_{i+1}$, assumption $\neg\Diamond a_{i+1}$ for the set of formulas $\neg a_{i+1}, \neg\Diamond a_{i+1}$. Then all assumptions with number $i+1$ may be eliminated.

2.3 Illustration

1. $A_1, ..., A_i, a_{i+1}, \neg\Diamond\neg a_{i+1} \Rightarrow D$,
2. $A_1, ..., A_i, a_{i+1}, \Diamond\neg a_{i+1} \Rightarrow D$,
3. $A_1, ..., A_i, \neg a_{i+1}, \Diamond a_{i+1} \Rightarrow D$,
4. $A_1, ..., A_i, \neg a_{i+1}, \neg\Diamond a_{i+1} \Rightarrow D$,
5. $A_1, ..., A_i, a_{i+1} \Rightarrow D$ – from 1, 2,
6. $A_1, ..., A_i, \neg a_{i+1} \Rightarrow D$ – from 3, 4,
7. $A_1, ..., A_i \Rightarrow D$ – from 5, 6.

In my doctoral thesis I brought forward 30 problems calling for solution. Later these ideas were published in monograph [8, p. 208–

217]. Many of these problems have been solved by now. The solutions were published in 13 PhD theses and publications. Some of the problems have not been solved yet. One of these problems (problem number 9) may be formulated as follows: *if logic is functionally complete, then for any propositional variable a and any truth value i there is a formula $f_i(a)$ containing only this variable and taking some designated value if and only if a takes value i; suppose $a_1, a_2, ..., a_n$ are all different variables occurring in D; suppose $b_1, b_2, ..., b_n$ are the truth-values of these variables; suppose A_s is $f_k(a_s)$, if b_s is k; suppose D' is $f_r(D)$ ($f_r(D)$ is a formula formed on the basis of D and taking designated value with truth-values $b_1, b_2, ..., b_n$ of the variables $a_1, a_2, ..., a_n$). Then $A_1, A_2, ..., A_n \Rightarrow D$.*

For example, 1, $\frac{1}{2}$, 0 are the truth-values of three-valued modal logic of Łukasiewicz; $f_1(a)$ is $\Box a$, $f_{\frac{1}{2}}(a)$ is $\Diamond a \& \Diamond \neg a$, $f_0(a)$ is $\neg \Diamond a$; if formula takes value 1 with some truth-values of its variables, then $f_r(D)$ is $\Box D$, etc.; assumptions may be eliminated like it was stated for Sb^-.

The ninth problem is the problem of finding the proof for metatheorem of semantic completeness of all known finite-valued matrix logics and finding sets of axioms for all logics of this kind stated semantically.

The seventeenth problem is the problem of generalization of this method for the proof of semantic completeness (and solution of the decision problem) of propositional quasi-matrix logics. This problem has not been solved for a long time. The solution is brought off in this article.

3 Quasi-matrix logic

Quasi-matrix is a set $(Q, G, qf_1, ..., qf_s)$, where Q and G are nonempty sets such that $Q \subseteq G$; $qf_1, ..., qf_s$ are quasi-functions.

If a function is a correspondence in virtue of which an object from some (functional) domain is related with certain object (from the range of the function) then a quasi-function is a correspondence in virtue of which an object from a certain subset of some set is related with some object from a certain subset of some or another set (from the range of the quasi-function).

3.1 Examples

Function: $\{(a,d),(b,k),(c,k)\}$.

Quasi-function: $\{(a,d)\veebar_2(a,k),(c,m)\} = \{\{(a,d),(c,m)\}\veebar_2\{(a,k),(c,m)\}\}$,

Quasi-function: $\{\veebar_4((a,k),(a,n),(c,k),(c,n)),(d,r) = \veebar_4[\{(a,k),(d,r)\},\{(a,n),(d,r)\},\{(c,k),(d,r)\},\{(c,n),(d,r)\}]\}$,

$\veebar_2$ and $\veebar_4$ are two- and four-place (respectively) metalinguistic exclusive disjunctions. Let us assume that disjunction may be degenerative, i. e. in this particular case quasi-function is just a function. Then a matrix is a particular case of quasi-matrix.

In the general case an object of application of a quasi-function, as well as truth-value of a quasi-function, are indefinite. Only subrange of the range of quasi-function, which includes this object, and sub-range of the range of values of a quasi-function, which contains a value of a quasi-function, are defined.

Such vagueness may be of a cognitive nature. It takes place, when the above-mentioned correspondence or relation is objectively functional, but this is not known to the researcher. For example, there are three probable variants of translation of a certain word in a dictionary, but the translator doesn't know, which of these three readings is the most appropriate in the present case (context). Such situations also appear in systems of automatic translation.

Another cause of indetermination is that reality may be indeterminate itself. For example, for planning of a production we have to take into account the following reasons. Suppose that we know the limits of alteration of a quantity of raw stuff, which will be factored next year. But it s impossible to figure out any rigid link between definite quantity of a factored raw stuff and a quantity of output, even if we knew a quantity of man-power, equipment etc.

For the first time some particular examples of quasi-functions were represented by H. Reichenbach (1932, 1935, 1936), Z. Zavarski (1936), F. Gonseth (1938, 1941), N. Rescher (1962, 1964, 1965, 1969). Rescher considers a material implication and defines it as follows:

A	$\supset$	B
t	t	t
t	f	f
f	(t, f)	t
f	(t, f)	f

(t, f) is not a determinate truth-values. This bracketed entry (t, f) means that either one of these two truth-values may occur in the various particular cases. Hence, depending on specific sense of propositions, the whole implication may be either true or false. Other logical terms are formulated in a usual way.

It is obvious that not all tautologies of a classical propositional logic of the form $A \supset B$ take the truth-value 't' under any given assignment of truth-values to elementary propositions.

Rescher formulates the conception of quasi-tautology. He adopts t and (t, f) in his quasi-functional system Q as designated truth-values. Then quasi-tautology is a formula which invariably does or can take either of this designated truth-values for every assignment of truth-values to its propositional variables. But if we bring to a logical end Rescher's reasoning we also have to treat as a quasi-tautology propositional variable p.

Then Rescher 'corrects' definitions of Łukasiewicz' three-valued logic.

$$\begin{array}{ccc} A & \& & B \\ \frac{1}{2} & (\frac{1}{2}, 0) & \frac{1}{2} \end{array}$$

Independently of the above-mentioned and some other authors I came to the same considerations at the end of the sixties / beginning of the seventies. My ideas were concerned with the way of modal logic development. Though by that time a lot of different 'logical systems' had been constructed, it wasn't clear, what kind of modal operators and notions (either factual or logical necessity, possibility etc.) were defined by these systems. It made the application of modal systems to the natural reasoning analysis very difficult. This condition of modal logic seemed to me unsatisfactory and inadequate. On purpose to overcome these difficulties I distinguished two different branches of modal logical investigations: proper logic (or logic itself) and an imitation of logic. Proper logic deals with the forms of thoughts. H. Curry called this kind of logic a philosophical one. Imitation of logic is a certain (formal) system, e. g. algebraic system, which in some respect resembles philosophical logic (usually with respect to some technical symbols and signs) [15].

In the following explanations I am treating modern logic as a philosophical logic in the sense of Curry.

In logic, as well as in each other science, it's possible to distinguish empirical and theoretical levels of development. An essential feature of a theory is its ability to explain phenomena. As I think, my approach to the analysis of logical modalities, elaborated by N. Arkhiereev, possesses this ability. Theory of factual modalities, which is to be based on quasi-matrix logic, has not been yet completely developed. (Fundamental ideas of theory of logical modalities are represented in [1, 2, 6, 7, 13, 14].)

I began to work out quasi-matrix logic with constructing the system of minimal modal logic.

3.2 Minimal modal logic S_{min}

(Symbols of formalised language: $\Box, \Diamond, \neg, \supset$).

Łukasiewicz's well-known statement about impossibility of proper definitions of modal operators 'necessary ($\Box$) and 'possibly' ($\Diamond$) in terms of 'truth' and 'falsity' is valid only if these operators are interpreted as functions.

But if we interpret modal operators as quasi-functions, it becomes possible to define them in above-mentioned terms.

Let's consider formula $\Box A$. Assume A takes value f (falsehood). Then formula $\Box A$ also takes value f, since not-existing state of affairs can not be necessary (both logically and factually). Assume formula A takes value t (truth). What truth-value takes formula $\Box A$ in this case? The value is indeterminate. Formula $\Box A$ takes either value t, or value f. Let's notify this situation by t/f.

By the same reasoning, we can conclude that truth-value of the formula $\Diamond A$ is indeterminate, when formula A takes value f. Definitions of signs of negation and implication are usual. Designated truth-value is t.

Principles of classical propositional logic and logic S_{min}

Classical propositional logic principles	**Principles of quasi-matrix logic S_{min}**
(1) the principle of bivalency (propositions take values from the domain $\{t$ (truth), f (falsity)$\}$)	the principle of bivalency
(2) the principle of consistency (a proposition can not have both the values)	the principle of consistency
(3) the principle of excluded middle (a proposition necessarily has some of these values)	the principle of excluded middle
(4) the principle of identity (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{t, f\}$)	the principle of identity
(5) the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in classical logic this principle acts as a matrix principle — logical connectives are interpreted as functions)	the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in S_{min} this principle acts as a quasi-matrix principle — logical terms are interpreted as quasi-functions)

S_{min} — formalism which is adequate to the system constructed semantically. S_{min}-calculus is an extension of a classical propositional calculus with added new axiom schemes: $\Box A \supset A$, $A \supset \Diamond A$.

S_{min}-calculus is weaker than basic modal logic of Łukasiewicz, since the formula $\Box A \equiv \neg\Diamond\neg A$ is not provable there.

For the proof of semantic completeness meta-theorem of S_{min}-calculus, we define alternative interpretation as follows.

Alternative interpretation is a function $|| \; ||$ such as to: If P is — propositional variable then $||P|| \in \{t, f\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = t \Leftrightarrow ||A|| = f$; $||A \supset B|| = f \Leftrightarrow ||A|| = f$ or $||B|| = t$; $||A|| = f \Rightarrow ||\Box A|| = f$; $||A|| = t \Rightarrow ||\Box A|| \in \{t, f\}$; $||A|| = t \Rightarrow ||\Diamond A|| = t$; $||A|| = f \Rightarrow ||\Diamond A|| \in \{t, f\}$. ($\Leftrightarrow$ and $\Rightarrow$ are here abbreviations for expression 'if and only if' ('iff') and 'if..., then...' respectively.)

Formula is satisfiable iff it takes the value 'true' in some alternative interpretation. Formula is valid iff it is true under each alternative interpretation.

3.3 Four-valued quasi-matrix logical systems

Truth-values t^n, t^c, f^c, f^i are interpreted as follows: proposition taking values t^n describes a state of affairs which takes place in reality and which is strictly determined by certain circumstances; proposition taking values t^c describes a state of affairs which takes place in reality and which is not strictly determined by either circumstances; proposition taking values f^c describes a state of affairs which doesn't exist in reality and the absence of which is not strictly determined by either circumstances; proposition taking values f^i describes a state of affairs which doesn't exist in reality and which absence is strictly determined by certain circumstances.

Four-valued quasi-matrix logic based on the following generalization of classical logic principles.

Classical logic principles	**Quasi-matrix logic principles**
(1) the principle of <u>bivalency</u> (propositions take values from the domain $\{t$ (truth), f (falsity)$\}$)	the principle of <u>four-valency</u> (propositions take values from the domain $\{t^n, t^c, f^c, f^i\}$)
(2) the principle of <u>consistency</u> (a proposition can not have both the values)	<u>consistency</u>: can not have more than one value from $\{t^n, t^c, f^c, f^i\}$
(3) the principle of <u>excluded middle</u> (a proposition necessarily has some of these values)	the principle of <u>excluded fifth</u>
(4) the principle of <u>identity</u> (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{t, f\}$)	<u>identity</u> from the domain $\{t^n, t^c, f^c, f^i\}$

(5) the principle of specifying the truth value of a complex proposition by truth values of elementary propositions constituting it (in propositional logic this principle acts as a matrix principle — logical connectives are defined by matrices, in predicate logic it shows up in the interpretation of logical terms and predicates as truth functions).	the quasi-matrix principle (logical terms are interpreted as quasi-functions)

Logical terms are the same as those in the S_{min}-system.

Definitions of logical terms:

A	$\neg A$	**a**		**b**		**c**		**d**		**e**	
		$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$
t^n	f^i	t	t	t^n	t^n	t^n	t^n	t^c	t^c	t^c	t^c
t^c	f^c	f	t	f^c	t^c	f^i	t^n	f^c	t^c	f^i	t^n
f^i	t^n	f	f	f^i	f^i	f^i	f^i	f^c	f^c	f^c	f^c
f^c	t^c	f	t	f^c	t^c	f^i	t^n	f^c	t^c	f^i	t^n

A	$\neg A$	**f**		**g**		**h**		**i**	
		$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$	$\Box A$	$\Diamond A$
t^n	f^i	t	t	t	t	t^n	t^n	t^c	t^c
t^c	f^c	f^i	t^n	f^c	t^c	f	t	f	t
f^i	t^n	f	f	f	f	f^i	f^i	f^c	f^c
f^c	t^c	f^i	t^n	f^c	t^c	f	t	f	t

(−)

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	t^c	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	t^c	t^c	t^c

( )

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	$t^n \mid t^c$	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	t^c	t^c	$t^n \mid t^c$

(+)

		B			
	$\supset$	t^n	t^c	f^i	f^c
A	t^n	t^n	t^c	f^i	f^c
	t^c	t^n	$t^n \mid t^c$	f^c	f^c
	f^i	t^n	t^n	t^n	t^n
	f^c	t^n	$t^n \mid t^c$	t^c	$t^n \mid t^c$

t and $t^n|t^c$ mean «either t^n, or t^c». f and $f^i|f^c$ mean «either f^i, or f^c».

Following logical systems have been constructed on the basis of above-stated definitions: Sa^-, Sa, Sa^+, Sb^-, Sb, Sb^+, Sc^-, Sc, Sc^+, Sd^-, Sd, Sd^+, Se^-, Se, Se^+, Sf^-, Sf, Sf^+, Sg^-, Sg, Sg^+, Sh^-, Sh, Sh^+, Si^-, Si, Si^+. Lower case letters occurring in the name of systems corresponds to the definition of modal terms, signs $+, -$ and their absence correspond to the definition of implication.

t^n and t^c are distinguished truth-values.

The following considerations underlie the above-stated definitions of logical terms. Let us consider formula $\Box\Box A$. If the subformula A takes value t, then the value of a formula $\Box A$, as it has already been settled, is not determined, i. e. situation which is described by A takes place in reality but is determined itself either strictly or not. In the first case we have to assign to the formula $\Box A$ value t, in the second one — the value f.

I.e. in the first case a proposition A is interpreted as being true and (factually) necessary (in our terms it takes value t^n). What value in this case takes formula $\Box\Box A$? If A describes a state of affairs which is strictly determined by any circumstances, then these circumstances may in its own turn be either determined or not by some others. That is formula $\Box A$ also takes value t^n (or t^c) etc.

Such situations occur both in subjective and objective reality.

Different kinds of distinct and fuzzy determination in biology were considered by V.Yu. Ivlev in [5,6].

Semantic-constructed systems are formalized by a number of calculi including as their general part all schemes of axioms of a classical propositional calculus, modus ponens — rule of inference and following schemes of axioms: $\Box A \supset A$; $\neg\Box\neg A \supset \Diamond A$; $\Diamond A \supset \neg\Box\neg A$; $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$; $\Diamond B \supset \Diamond(A \supset B)$; $\Diamond\neg A \supset \Diamond(A \supset B)$; $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.

We sign with letter S the calculus, which is obtained from classic propositional calculus by means of above-stated eight model schemes of axioms. The calculi corresponding to the semantic-constructed systems may be worked out by addition to S of the following schemes of axioms:

Sa^-: $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$.

Sa: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$.

Sa^+: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$.

Sb^-: $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Sb: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Sb^+: $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Box\Diamond A$; $\Box\Diamond A \supset \Box A$; $\Diamond\Diamond A \supset \Diamond A$.

Calculi $Sc^-, Sd^-, Se^-, Sf^-, Sg^-, Sh^-, Si^-$ include schemes of axioms $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$.

Calculi $Sc, Sd, Se, Sf, Sg, Sh, Si$ include schemes of axioms $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))$.

Calculi $Sc^+, Sd^+, Se^+, Sf^+, Sg^+, Sh^+, Si^+$ include schemes of axioms $\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$;.

Calculi, which have the same lower case letter occurring in the names (e. g. calculi Sc^-, Sc, Sc^+), differ from calculi, which have other lower case letters occurring in the names (e. g. calculi Si^-, Si, Si^+), by sets of schemes of axioms $\{\Box(A \supset B) \supset (\Diamond A \supset \Box B)\}$, $\{\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$; $\Box(A \supset B) \supset (\Diamond A \supset (\Diamond\neg B \supset (\neg A \supset \neg B)))\}$, $\{\Box(A \supset B) \supset (\Box A \supset \Box B)$; $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)\}$.

The other additional schemes of axioms of these calculi are the same:

Calculi Sc^-, Sc, Sc^+: $\Box A \supset \Box\Box A$; $\Diamond\Diamond A \supset \Diamond A$; $\Diamond\Box A \supset \Box A$; $\Diamond A \supset \Box\Diamond A$.

Calculi Sd^-, Sd, Sd^+: $\Diamond A^*$, A^* is modalized formula.

Calculi Se^-, Se, Se^+: $\Diamond\Diamond A$; $\Diamond\neg\Box A$; $\neg\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Diamond\neg\Diamond A$; $\Diamond\Box A \supset (A \supset \Box A)$; $\Diamond\Box A \supset (\Diamond A \supset A)$; $A \supset (\Diamond\neg A \supset \Box\Diamond A)$; $\neg A \supset (\Diamond A \supset \Box\Diamond A)$.

Calculi Sf^-, Sf, Sf^+: $\Diamond\Box A \supset (A \supset \Box A)$; $\Diamond\Box A \supset (\Diamond A \supset A)$; $A \supset (\Diamond\neg A \supset \Box\Diamond A)$; $\neg A \supset (\Diamond A \supset \Box\Diamond A)$.

Calculi Sg^-, Sg, Sg^+: $A \supset (\neg\Box A \supset \Diamond\Box A)$; $\neg A \supset (\Diamond A \supset \Diamond\Box A)$; $\Box\Diamond A \supset (A \supset \Box A)$; $\Box\Diamond A \supset (\Diamond A \supset A)$.

Calculi Sh^-, Sh, Sh^+: $\Box A \supset \Box\Box A$; $\Diamond\Box A \supset \Diamond A$; $\Box A \supset \Box\Diamond A$; $\Diamond\Diamond A \supset \Diamond A$.

Calculi Si^{-}, Si, Si^{+}: $\Diamond\Diamond A$; $\Diamond\neg\Box A$; $\neg\Diamond A \supset \Diamond\Box A$; $\Box A \supset \Diamond\neg\Diamond A$.

We use the rule of substitution of $\neg\neg A$ with A and visa versa.

For the proof of metatheorem of semantic completeness of calculi $Sb^{-}, Sc^{-}, Sd^{-}, Se^{-}$ (semantics for these calculi are of matrix sort) the following **lemma** is proved.

LEMMA 3. Assuming that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables. Let A_i be $\Box a_i$, $a_i \& \Diamond\neg a_i$, $\neg\Diamond a_i$, $\neg a_i \& \Diamond a_i$ depending on whether b_i is t^n, t^c, f^i or f^c. Let D' be $\Box D$, $D \& \Diamond\neg D$, $\neg\Diamond D$ or $\neg D \& \Diamond D$ depending on whether D takes value t^n, t^c, f^i or f^c with truth-values $b_1, ..., b_n$ variables $a_1, ..., a_n$. Then $A_1, ..., A_n \Rightarrow D'$.($\Rightarrow$ is here a sign for entailment.)

Lemma is proved by the use of recurrent mathematical induction.

Semantics for others calculi are quasi-matrix (proper). For the proof of metatheorem of semantic completeness of these calculi the notion of alternative interpretation is used. We have the following definition of alternative interpretation for Sa^{+}-system.

Alternative interpretation is a function $|| \; ||$ satisfying the following:

If P is — propositional variable then $||P|| \in \{t^n, t^c, f^i, f^c\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = t^n \Leftrightarrow ||A|| = f^i$; $||\neg A|| = t^c \Leftrightarrow ||A|| = f^c$; $||\neg A|| = f^i \Leftrightarrow ||A|| = t^n$; $||\neg A|| = f^c \Leftrightarrow ||A|| = t^c$;

$||A \supset B|| = f^c \Leftrightarrow (||A|| = t^n$ and $||B|| = f^c)$ or $(||A|| = t^c$ and $||B|| = f^i)$;

$||A \supset B|| = f^i \Leftrightarrow ||A|| = t^n$ and $||B|| = f^i$;

if either $(||A|| = t^n$ and $||B|| = t^c)$ or $(||A|| = f^c$ and $||B|| = f^i)$, then $||A \supset B|| = t^c$;

if $||A|| = f^i$ or $||B|| = t^n$, then $||A \supset B|| = t^n$;

if either $||A|| = ||B|| = t^c$ or $(||A|| = f^c$ and $||B|| = t^c)$, or $||A|| = ||B|| = f^c)$, then $||A \supset B|| \in \{t^n, t^c\}$;

$||A|| = t^n \Rightarrow ||\Box A|| \in \{t^n, t^c\}$; if either $||A|| = t^c$ or $||A|| = f^c$, or $||A|| = f^i$, then $||\Box A|| \in \{f^c, f^i\}$;

$||A|| = f^i \Rightarrow ||\Diamond A|| \in \{f^c, f^i\}$; if either $||A|| = t^n$ or $||A|| = t^c$, or $||A|| = f^c$, then $||\Diamond A|| \in \{t^n, t^c\}$.

S_r — three-valued quasi-matrix logic.

(Symbols of formalised language are the same.) n, c, i — values of S_r-system — which are interpreted respectively as 'necessary', 'contingently', 'impossibly'. State of affairs is necessary if and only if (iff) it is distinctly determined by certain circumstances; state of affairs is contingent, iff neither its existence nor its absence is not strictly determined by some circumstances; state of affairs is impossible iff its absence is strictly determined by some circumstances. Actually, here and above the evaluations of state of affairs concern (to) propositions. (To my regret, I couldn't find proper terms for evaluation of propositions.)

S_r-logic is based on the following generalizations of principles of classic logic.

Classical logic principles	**Principles of quasi-matrix logic S_r**
(1) the principle of bivalency	the principle of three-valency (propositions take values from the domain $\{n, c, i\}$)
(2) the principle of consistency	consistency: can not have more than one value from $\{n, c, i\}$
(3) the principle of excluded middle	the principle of excluded fourth
(4) the principle of identity	Identity (in a complex proposition, a system of propositions, an argument one and the same proposition has one and the same value from the domain $\{n, c, i\}$)
(5) the matrix principle	the quasi-matrix principle (logical terms are interpreted as quasi-functions)

Definitions of logical terms:

A	$\neg A$	$\Box A$	$\Diamond A$
n	i	n	n
c	c	i	n
i	n	i	i

$\supset$	n	c	i
n	n	c	i
c	n	$n\|c$	c
i	n	n	n

$n|c$ is interpreted as 'either n or c'. n is a designated value.

Corresponding calculus includes all schemes of axioms of classical propositional calculus (note: in these schemes of axioms metasymbols A, B, C denote modalized formulas; the modalized formula definition: if A is a formula of classical propositional calculus, then $\Box A$ and $\Diamond A$ are modalized formulas; if B and C are modalized formulas, then $\Box B$, $\Diamond B$, $\neg B$, $(B\&C)$, $(B \vee C)$, $(B \supset C)$ are modalized formulas; nothing else is a modalized formula.), modus ponens, Godel's rule, all schemes of axioms of Sc^{+}-calculus, and besides the following schemes: $\Box A \supset \Diamond A$; $\neg A \supset \neg\Box A$; $\neg\Diamond A \supset \neg A$; $A \supset \Diamond A$.

Alternative interpretation is a function $|| \ ||$ for which the following helds:

If P is propositional variable then $||P|| \in \{n, c, i\}$.

If $||A||$ and $||B||$ are defined, then $||\neg A|| = n \Leftrightarrow ||A|| = i$; $||\neg A|| = c \Leftrightarrow ||A|| = c$; $||\neg A|| = i \Leftrightarrow ||A|| = n$;

if either $||A|| = i$ or $||B|| = n$, then $||A \supset B|| = n$;

if $||A|| = ||B|| = c$, then $||A \supset B|| \in \{n, c\}$;

if either $\{||A|| = c$ and $||B|| = i\}$ or $\{||A|| = n$ and $||B|| = c\}$, then $||A \supset B|| = c$;

$||A|| = n$ and $||B|| = i$, iff $||A \supset B|| = i$;

$||\Box A|| = n$ iff $||A|| = n$; $||\Box A|| = i$, iff {either $||A|| = c$ or $||A|| = i$};

$||\Diamond A|| = i$, iff $||A|| = i$; $||\Diamond A|| = n$, iff {either $||A|| = n$ or $||A|| = c$}.

The formalisation and the proof of the meta-theorem of semantic completeness are the same as they were stated above.

3.4 Some peculiar properties of this logical system

First of all, it allows the use of the rule $A \Rightarrow \Box A$.

Besides, all derivable rule of inference of a classical propositional calculus are applicable to modalized formulas only. Some (at least some) direct rules of inference of a classical propositional calculus are also applicable to non-modalized formulas, for example: $A \vee B, \neg A \Rightarrow B$; but such indirect rules as rule of deduction:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

and rule reductio ad absurdum

$$\frac{\Gamma, A \Rightarrow B; \Gamma, A \Rightarrow \neg B}{\Gamma \Rightarrow \neg A}$$

are not applicable to non-modalized formulas in derivation.

However, so-called weakened rule of reductio ad absurdum

$$\frac{\Gamma, A \Rightarrow B; \Gamma, A \Rightarrow \neg B}{\Gamma \Rightarrow \Diamond\neg A}$$

is applicable to any formula in derivation.

4 Generalisation for quasimatrix logic

4.1 For logic S_{min}

LEMMA 4. suppose that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables; let A_i be a_i or $\neg a_i$, depending on whether b_i is t or f; let D' be D or $\neg D$ depending on whether D takes value t or f with truth-values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in every alternative interpretation, formed on the basis of some initial interpretation. Let D' be $D \vee \neg D$ depending on whether D takes value t under the truth assignment $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in some alternative interpretation formed on the basis of the initial interpretation, or it takes value f under the truth assignment $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in some alternative interpretation formed on the basis of the initial interpretation. Then $A_1, ..., A_n \Rightarrow D'$.

If in some alternative interpretations formula D takes value t and in some alternative interpretations it takes value f, then statement '$A_1, ..., A_n \Rightarrow D \vee \neg D$' may be substituted for the statement '$A_1, ..., A_n \Rightarrow D$ or $A_1, ..., A_n \Rightarrow \neg D$'.

PROOF. Lemma is proved by the use of recurrent mathematical induction.

Basis of induction. D does not contain any logical terms. Proof is obvious.

Assumption of induction. Proof holds for the formulas, containing k ($k \leq n$) occurrences of logical terms.

Step of induction. Proof holds for the formulas containing $n+1$ occurrences of logical terms.

Case 1. $n+1$-th occurrence of the logical terms is the occurrence of the sign of negation. Formula D is $\neg B$.

Suppose formula D takes value t in all alternative interpretations, formed on the basis of some initial interpretation. Then B takes value f in all these alternative interpretations. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg B$.

Suppose formula D takes value f in all alternative interpretations, formed on the basis of some initial interpretation. Then B takes value t in all these alternative interpretations and by the assumption of induction $A_1, ..., A_n \Rightarrow B$. Then $A_1, ..., A_n \Rightarrow \neg\neg B$.

Under the third possibility $A_1, ..., A_n \Rightarrow \neg B \vee \neg\neg B$.

Case 2. $n+1$-th occurrence of the logical terms is the occurrence of the sign of necessity. Formula D is $\Box B$. Suppose B takes value f in all alternative interpretations, formed on the basis of some initial interpretation. Then by the assumption of induction $A_1, ..., A_n \Rightarrow \neg B$. Since $\neg B \supset \neg\Box B$ is a theorem scheme (contraposition of axiom scheme $\Box B \supset B$), then $A_1, ..., A_n \Rightarrow \neg\Box B$. If B takes value t in all or some alternative interpretations, then formula $\Box B$ takes value t in some alternative interpretations and in some other alternative interpretations it takes value f. Then it is obvious that $A_1, ..., A_n \Rightarrow \Box B \vee \neg\Box B$.

Case 3. $n+1$-th occurrence of the logical terms is the occurrence of the sign of possibility. Formula D is $\Diamond B$. Suppose B takes value t in all alternative interpretations, formed on the basis of some initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow B$. Since $B \supset \Diamond B$ is a theorem , $A_1, ..., A_n \Rightarrow \Diamond B$. If B takes value f in all or some alternative interpretations, then formula $\Diamond B$ takes value t in some alternative interpretations and it takes value f in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow \Diamond B \vee \neg\Diamond B$.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$. If formula D under above-mentioned truth-assignments of its variables takes value t in some alternative interpretations and in some other alternative interpretations it takes value f, then D' is $(B \supset C) \vee \neg(B \supset C)$. The entailment is obvious. If D takes value f, then D' is $\neg(B \supset C)$.

It is possible if in every alternative interpretation formula B takes value t and formula C takes value f. By the assumption of induction for every alternative interpretation holds that $A_1, ..., A_n \Rightarrow B$ and $A_1, ..., A_n \Rightarrow \neg C$. Consequently $A_1, ..., A_n \Rightarrow \neg(B \supset C)$. Let's take into consideration the last case, then D takes value t in every alternative interpretation. It means that in every alternative interpretation formula B takes value f or formula C takes value t. Hence by the assumption of induction,

$$A_1, ..., A_n \Rightarrow \neg B$$
or
$$A_1, ..., A_n \Rightarrow C.$$

Analyzing all possible cases we conclude: $A_1, ..., A_n \Rightarrow (B \supset C)$. □

4.2 For logic S_r

LEMMA 5. Suppose that D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are values of these variables; let A_i be $\Box a_i$, $\Diamond a_i \& \Diamond \neg a_i$, $\neg \Diamond a_i$, depending on whether b_i is n, c, or i. Let D' be $\Box D$, $\Diamond D \& \Diamond \neg D$ or $\neg \Diamond D$, depending on whether D takes value n, c, or i with values $b_1, ..., b_n$ variables $a_1, ..., a_n$ in all alternative interpretations, formed on the basis of some initial interpretation; suppose D' is $\Box D \vee (\Diamond D \& \Diamond \neg D)$, $\Box D \vee \neg \Diamond D$, $(\Diamond D \& \Diamond \neg D) \vee \neg \Diamond D$, $(\Box D \vee (\Diamond D \& \Diamond \neg D)) \vee \neg \Diamond D$, depending on whether D takes, respectively, value n in some alternative interpretations and in some other alternative interpretations it takes value c; D takes value n in some alternative interpretations and in some others it takes value i; D takes value c in some alternative interpretations and in some others it takes value i; D takes value n in some alternative interpretations or it takes value c in some other alternative interpretations, or it takes value i in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow D'$.

If D' is $\Box D_i \vee (\Diamond D_i \& \Diamond \neg D_i)$, statement '$A_1, ..., A_n \Rightarrow D'$' may be substituted for '$A_1, ..., A_n \Rightarrow \Box D_i$ or $A_1, ..., A_n \Rightarrow \Diamond D_i \& \Diamond \neg D_i$'. The substitution of the same kind is possible in case of other values in different alternative interpretations. I.e, logical entailment is based on alternative interpretations formed on the basis of some initial interpretation. For example, if formula takes value n in every alternative interpretation, then the following holds for these alternative interpretations '$A_1, ..., A_n \Rightarrow \Box D_i$ or $A_1, ..., A_n \Rightarrow \Box D_i$, or $A_1, ..., A_n \Rightarrow \Box D_i$'. Hence $A_1, ..., A_n \Rightarrow \Box D_i$. Note that if there is no any ambiguity the only alternative interpretation that is possible is the initial one. In this case $A_1, ..., A_n \Rightarrow \Box D_i$ also holds. The same holds for the other values.

PROOF. Lemma is proved by recurrent mathematical induction on the number of occurrences of logical terms in formula D.

Step of induction.

Case 1. Formula D is $\neg B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$. $\neg\Diamond B \supset \Box\neg B$ is a theorem scheme. (Using theorem scheme $\neg\Box\neg A \supset \Diamond A$.) Then $A_1, ..., A_n \Rightarrow \Box\neg B$.

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value n in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond\neg B$. Here we use the axiom scheme $\Diamond A \supset \neg\Box\neg A$ and the rule of substitution of $\neg\neg A$ for A and vice versa.

Suppose D takes value c in every alternative interpretation formed on the basis of some initial interpretation. Then B also takes value c in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$. Hence $A_1, ..., A_n \Rightarrow (\Diamond\neg B \& \Diamond\neg\neg B)$.

Suppose D takes value n in some alternative interpretations and it takes value c in some others. By the assumption of induction: $A_1, ..., A_n \Rightarrow \neg\Diamond B$ or $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$. Since in the first case $A_1, ..., A_n \Rightarrow \Box\neg B$ and in the second one $A_1, ..., A_n \Rightarrow (\Diamond\neg B \& \Diamond\neg\neg B)$, the following holds: $A_1, ..., A_n \Rightarrow \Box\neg B \vee (\Diamond\neg B \& \Diamond\neg\neg B)$.

For other possible cases proof is analogous.

Case 2. Formula D is $\Box B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B also takes value n in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \Box\Box B$. (Using axiom scheme $\Box A \supset \Box\Box A$.)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation, or it takes value c in every alternative interpretation, or it takes value i in some alternative interpretation and it takes value c in some another alternative interpretation. Under the last possibility by the assumption of induction

$$A_1, ..., A_n \Rightarrow \neg\Diamond B$$
or
$$A_1, ..., A_n \Rightarrow (\Diamond B \& \Diamond\neg B).$$

In both cases $A_1, ..., A_n \Rightarrow \neg\Diamond\Box B$. (In the first case we use axioms schemes $\Diamond\Box A \supset \Box A$ and $\Box A \supset \Diamond A$, and in second one – $\Diamond\Box A \supset \Box A$ and $\Diamond A \supset \neg\Box\neg A$.) Formula D can not take value c.

If formula D takes different truth values in different alternative interpretations the proof may be concluded from the above-analyzed cases.

Case 3. Formula D is $\Diamond B$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value n in every alternative interpretation, or it takes value c in every alternative interpretation, or it takes value n in some alternative interpretation and it takes value c in another alternative interpretation. Under the last possibility by the assumption of induction

$$A_1, ..., A_n \Rightarrow \Box B$$
or
$$A_1, ..., A_n \Rightarrow (\Diamond B \& \Diamond\neg B).$$

In both cases $A_1, ..., A_n \Rightarrow \Box\Diamond B$. (In the first case we use axioms schemes $\Box A \supset \Diamond A$ and $\Diamond A \supset \Box\Diamond A$, and in the second case we need only the last axiom)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value i in every alternative interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B$. (Using the axiom scheme $\Diamond\Diamond A \supset \Diamond A$.) Formula D can not take value c. If formula D takes different truth values in different alternative interpretations the proof may be concluded from the above-analyzed cases.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$.

Suppose D takes value n in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value i in every alternative interpretation or C takes value n in every alternative interpretation. By the assumption of induction for every alternative interpretation holds: $A_1, ..., A_n \Rightarrow \neg\Diamond B$ or $A_1, ..., A_n \Rightarrow \Box C$. Hence: $A_1, ..., A_n \Rightarrow \Box(B \supset C)$. (Using axiom schemes $\neg\Diamond A \supset \Box(A \supset B)$; $\Box B \supset \Box(A \supset B)$.)

Suppose D takes value i in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value n in every alternative interpretation and C takes value i in every alternative interpretation. By the assumption of induction for every alternative interpretation holds: $A_1, ..., A_n \Rightarrow \Box B$ и $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Then $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. (Using axiom schemes $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.)

Suppose D takes value c in every alternative interpretation formed on the basis of some initial interpretation. It is possible if B takes value n and C takes value c in every alternative interpretation or B takes value c and C takes value i in every alternative interpretation. In the first case $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow \Diamond C \& \Diamond\neg C$. Then we have to prove: $A_1, ..., A_n \Rightarrow \Diamond(B \supset C) \& \Diamond\neg(B \supset C)$.

$A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using theorem scheme $\Diamond B \supset \Diamond(A \supset B)$). $A_1, ..., A_n \Rightarrow \Diamond\neg(B \supset C)$ (using axiom schemes $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and $\neg\Box\neg A \supset \Diamond A$, and rule of substitution of $\neg\neg A$ for A and vice versa). In second case $A_1, ..., A_n \Rightarrow \Diamond B \& \Diamond\neg B$, and $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Then $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using axiom scheme $\Diamond\neg B \supset \Diamond(A \supset B)$). $A_1, ..., A_n \Rightarrow \Diamond\neg(B \supset C)$ (using axiom schemes $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ and $\neg\Box\neg A \supset \Diamond A$).

Suppose D takes value n in some alternative interpretation formed on the basis of some initial interpretation and it takes value c in another interpretation. Then we have to prove: $A_1, ..., A_n \Rightarrow$

$\Diamond(B \supset C)\&\Diamond\neg(B \supset C)$ or $A_1, ..., A_n \Rightarrow \Box(B \supset C)$, or the equivalent statement $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$. This case is possible if both B and C takes value c in all alternative interpretations. By the assumption of induction $A_1, ..., A_n \Rightarrow \Diamond B\&\Diamond\neg B$ and $A_1, ..., A_n \Rightarrow \Diamond C\&\Diamond\neg C$. Then $A_1, ..., A_n \Rightarrow \Diamond(B \supset C)$ (using axiom scheme $\Diamond B \supset \Diamond(A \supset B)$).

The proof of other possibilities may be concluded from the above-analyzed cases. □

METATHEOREM 1. *If formula D is universally satisfiable then it is provable.*

Since for every truth-assignment of the variables holds $A_1, ..., A_n \Rightarrow \Box D$, then the following holds:

1. $A_1, ..., A_{n-1}, \Box a_n \Rightarrow \Box D$,
2. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow \Box D$,
3. $A_1, ..., A_{n-1}, \Diamond a_n\&\Diamond\neg a_n \Rightarrow \Box D$.

   Hence:

4. $A_1, ..., A_{n-1}, \Diamond a_n, \neg\Diamond\neg a_n \Rightarrow \Box D$, from 1,
5. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow \Box D$, from 2,
6. $A_1, ..., A_{n-1}, \Diamond a_n, \Diamond\neg a_n \Rightarrow \Box D$, from 3.
7. $A_1, ..., A_{n-1}, \Diamond a_n \Rightarrow \Box D$, from 4, 6,
8. $A_1, ..., A_{n-1} \Rightarrow \Box D$, from 5, 7. etc.

As $\Box D$ entails D, D is provable.

REMARK 1. Since formula can take one of the seven values (n, c, i, n/c, n/i, c/i, $n/c/i$), the problem arises to construct 7-valued logic with this values (lets sign them with 1, 2, 3, 4, 5, 6, 7) and compare it with S_r.

4.3 For logic Sa-

LEMMA 6. Suppose D is a formula, $a_1, ..., a_n$ are all different variables, occurring in D, $b_1, ..., b_n$ are truth-values of these variables; let A_i be $\Box a_i$, $a_i \& \Diamond \neg a_i$, $\neg \Diamond a_i$, $\neg a_i \& \Diamond a_i$, depending on whether b_i is t^n, t^c, f^i or f^c. Let D' be $\Box D$, $D \& \Diamond \neg D$, $\neg \Diamond D$ or $\neg D \& \Diamond D$, depending on whether D takes value t^n, t^c, f^i or f^c with values $b_1, ..., b_n$ of the variables $a_1, ..., a_n$ in all alternative interpretations formed on the basis of some initial interpretation. Suppose D' is $\Box D \vee (D \& \Diamond \neg D)$, $\Box D \vee \neg \Diamond D$, $(D \& \Diamond \neg D) \vee \neg \Diamond D$, $(\Box D \vee (D \& \Diamond \neg D)) \vee \neg \Diamond D$ and so on, depending on whether D takes respectively value t^n in some alternative interpretations and in some other alternative interpretations it takes value t^c; D takes value t^n in some alternative interpretations and in some others it takes value f^i; D takes value t^c in some alternative interpretations and in some others it takes value f^i; D takes value t^n in some alternative interpretations or it takes value t^c in some other alternative interpretations, or it takes value f^i in some other alternative interpretations. Then $A_1, ..., A_n \Rightarrow D'$.

PROOF. Lemma is proved by recurrent mathematical induction on the number of occurrence of logical terms in formula D.

Step of induction.

Case 1. Formula D is $\neg B$.

Suppose D takes value t^n in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^i in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg \Diamond B$. $\neg \Diamond B \supset \Box \neg B$ is a theorem scheme. (Using theorem scheme $\neg \Box \neg A \supset \Diamond A$.) Then $A_1, ..., A_n \Rightarrow \Box \neg B$.

Suppose D takes value f^i in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then $A_1, ..., A_n \Rightarrow \neg \Diamond \neg B$. Here we use an axiom scheme $\Diamond A \supset \neg \Box \neg A$ and rule of substitution of $\neg \neg A$ for A and vice versa.

Suppose D takes value t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^c in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$. Hence $A_1, ..., A_n \Rightarrow \neg B \& \Diamond \neg\neg B$.

Suppose D takes value f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^c too in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow B \& \Diamond \neg B$. Hence $A_1, ..., A_n \Rightarrow \neg\neg B \& \Diamond \neg B$.

Suppose D takes value t^n in some alternative interpretations formed on the basis of some initial interpretation and it takes value t^c in some other interpretations. By the assumption of induction B takes value f^i in some alternative interpretations and it takes value f^c in other alternative interpretations. Then $A_1, ..., A_n \Rightarrow \neg \Diamond B$ or $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$.

Since in the first case $A_1, ..., A_n \Rightarrow \Box \neg B$ and in the second $A_1, ..., A_n \Rightarrow \neg B \& \Diamond \neg\neg B$, the following holds: $A_1, ..., A_n \Rightarrow \Box \neg B \vee (\neg B \& \Diamond \neg\neg B)$.

For other possible cases proof is analogous.

Case 2. Formula D is $\Box B$.

Suppose D takes value t^n or t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n in every alternative interpretation formed on the basis of this initial interpretation. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$. Then we have to prove: $A_1, ..., A_n \Rightarrow \Box\Box B \vee (\Box B \& \Diamond \neg \Box B)$.

$\Box\Box B \vee (\Box B \& \Diamond \neg \Box B) \Leftrightarrow (\Box\Box B \vee \Box B) \& (\Box\Box B \vee \Diamond \neg \Box B)$.

$(\Box\Box B \vee \Box B) \& (\Box\Box B \vee \Diamond \neg \Box B) \Leftrightarrow (\Box\Box B \vee \Box B) \& (\Box\Box B \vee \neg \Box\Box B)$.

$(\Box\Box B \vee \Box B) \& (\Box\Box B \vee \neg \Box\Box B) \Leftrightarrow \Box B$.

Proof is completed. ($\Leftrightarrow$ is a sign for metalanguage equivalence).

Suppose D takes value f^i or f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^c or f^i, or f^c in every alternative interpretation formed on the basis of this initial interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \neg \Diamond \Box B \vee (\neg \Box B \& \Diamond \Box B)$. That is, we have to prove: $A_1, ..., A_n \Rightarrow \neg \Box B$.

In the first case by the assumption of induction $A_1, ..., A_n \Rightarrow B \& \Diamond \neg B$. The proof is evident.

In the second case $A_1, ..., A_n \Rightarrow \neg\Diamond B$. $\neg\Diamond B \Rightarrow \neg\Box B$. (Using axiom schemes $\neg\Box\neg A \supset \Diamond A$ and $\Box A \supset A$.) The statement is proved.

In the third case $A_1, ..., A_n \Rightarrow \neg B \& \Diamond B$. $\neg B \Rightarrow \neg\Box B$. The statement is proved.

Case 3. Formula D is $\Diamond B$.

Suppose D takes value t^n or t^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value t^n or t^c, or f^c in every alternative interpretation formed on the basis of this initial interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \Box\Diamond B \vee (\Diamond B \wedge \Diamond\neg\Diamond B)$. That is we have to prove: $A_1, ..., A_n \Rightarrow \Diamond B$. By the assumption of induction in every of three cases $A_1, ..., A_n \Rightarrow \Diamond B$.

Suppose D takes value f^i or value f^c in every alternative interpretation formed on the basis of some initial interpretation. Then B takes value f^i in every alternative interpretation. We have to prove: $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B)$. By the assumption of induction $A_1, ..., A_n \Rightarrow \neg\Diamond B$.

$\neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B) \Leftrightarrow \neg\Diamond B$

So $A_1, ..., A_n \Rightarrow \neg\Diamond\Diamond B \vee (\neg\Diamond B \& \Diamond\Diamond B)$ is proved.

Cases when formula D takes different values in different alternative interpretations may be reduced to the above-analyzed cases.

Case 4. $n+1$-th occurrence of the logical terms is the occurrence of the sign of implication. Formula D is $B \supset C$.

Suppose formula D takes value t^n in every alternative interpretation. It is possible if either B takes value f^i or C takes value t^n. We have to prove: $A_1, ..., A_n \Rightarrow \Box(B \supset C)$. The statement may be easily proved by axiom schemes $\neg\Diamond A \supset \Box(A \supset B)$, $\Box A \supset \Box(A \supset B)$.

Suppose formula D takes value f^i in every alternative interpretation. Then B takes value t^n and C takes value f^i. We have to prove: $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. By the assumption of induction $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow \neg\Diamond C$. Hence, $A_1, ..., A_n \Rightarrow \neg\Diamond(B \supset C)$. (Using axiom scheme $\Diamond(A \supset B) \supset (\Box A \supset \Diamond B)$.)

Suppose formula D takes value t^c in every alternative interpretation. It is possible if both B and C takes value t^c in every alternative interpretation, or if B takes value t^n and C takes value t^c, or if B takes value f^c and C takes one of the three values: t^c or f^i or f^c.

We have to prove: $A_1, ..., A_n \Rightarrow (B \supset C)\&\Diamond\neg(B \supset C)$. Under the first condition $A_1, ..., A_n \Rightarrow B\&\Diamond\neg B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$. $C \Rightarrow B \supset C$. $B \Rightarrow \Diamond B$. $\Diamond\neg C \Rightarrow \neg\Box C$. $\Diamond B\&\neg\Box C \Rightarrow \Diamond\neg(B \supset C)$. (Using axiom schemes $\Box(A \supset B) \supset (\Diamond A \supset \Box B)$, $\neg\Box\neg A \supset \Diamond A$.)

Under the second condition $A_1, ..., A_n \Rightarrow \Box B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$. The proof is the same as in the previous case.

Under the third condition $A_1, ..., A_n \Rightarrow \neg B\&\Diamond B$ and $A_1, ..., A_n \Rightarrow C\&\Diamond\neg C$ or $A_1, ..., A_n \Rightarrow \neg\Diamond C$, or $A_1, ..., A_n \Rightarrow \neg C\&\Diamond C$. In any case $A_1, ..., A_n \Rightarrow \neg\Box C$. The proof is completed. $\Box$

Cases when formula D takes different values in different alternative interpretations may be reduced to the above-analyzed cases.

METATHEOREM 2. *If formula D is universally satisfiable then it is provable.*

(Since for every truth-assignment of the variables holds $A_1, ..., A_n \Rightarrow \Box D$ or $A_1, ..., A_n \Rightarrow (D\&\Diamond\neg D)$ then the following holds: $A_1, ..., A_n \Rightarrow D$.)

1. $A_1, ..., A_{n-1}, \Box a_n \Rightarrow D$,
2. $A_1, ..., A_{n-1}, \neg\Diamond a_n \Rightarrow D$,
3. $A_1, ..., A_{n-1}, a_n\&\Diamond\neg a_n \Rightarrow D$,
4. $A_1, ..., A_{n-1}, \neg a_n\&\Diamond a_n \Rightarrow D$,

   Hence

5. $A_1, ..., A_{n-1}, \neg\Diamond\neg a_n \Rightarrow D$, from 1,
6. $A_1, ..., A_{n-1}, \neg a_n, \neg\Diamond a_n \Rightarrow D$, from 2,
7. $A_1, ..., A_{n-1}, a_n, \Diamond\neg a_n \Rightarrow D$, from 3,
8. $A_1, ..., A_{n-1}, \neg a_n, \Diamond a_n \Rightarrow D$, from 4,

   And then:

9. $A_1, ..., A_{n-1}, a_n \Rightarrow D$, from 5, 7,
10. $A_1, ..., A_{n-1}, \neg a_n \Rightarrow D$, from 6, 8,
11. $A_1, ..., A_{n-1} \Rightarrow D$, from 9, 10, and so forth.

References

[1] Arkhiereev, N., Semantics of restricted sets of state-descriptions for propositional logic, *Bulletin of M.S.U., ser. 'Philosophy'* 5:44–57, 1993 (in Russian).

[2] Arkhiereev, N., Abstracts of PHD, Thesis 'Semantics of restricted sets of state-descriptions', Moscow, 2001 (in Russian).

[3] Curry, H. B., *Foundations of Mathematical Logic*, New York, San Francisco, Toronto, London, 1963.

[4] Finn, V. K., On some characteristic Truth-tablaus of classical logic and three valued Łukasiewicz logic, in *Investigations of logical systems*, Moscow, 1970, pp. 215–261 (in Russian).

[5] Ivlev, V. Y., 'Necessity', 'Contingency', 'Possibility' in biology and their philosophical generalizations, *Categories. Philosophical Journal* 2:22–42, 1997 (in Russian).

[6] Ivlev, V. Y., Categories of necessity, contingency and possibility: their sense and methodological functions in scientific cognition, *Philosophy and society* 2, 1997 (in Russian).

[7] Ivlev, V. Y., and Y. V. Ivlev, Problems of construction of theory of factual modalitis, *Logical investigations* 7:269–278, 2000 (in Russian).

[8] Ivlev, Y. V., *Modal logic*, Moscow, 1991, 224 p. (in Russian).

[9] Ivlev, Y. V., *Contentive semantic of modal logic*, Moscow, 1985, 170 p. (in Russian).

[10] Ivlev, Y. V., Quasi-functional logic, *Scientific and technical information. Ser. 2. Inform. processes and systems* 6, 1992 (in Russian).

[11] Ivlev, Y. V., Truth-tables for modal logic, *Bulletin of M.S.U., ser. 'Philosophy'* 6, 1973 (in Russian).

[12] Ivlev, Y. V., Theory of Logical Modalities, *Multi. Val. Logic*, 5:91–102, 2000.

[13] Ivlev, Y. V., Outlines of the transition from the principles of traditional logic to the principles of non-classical logic, in *Zwischen traditioneller und modernen logik. Nichnklassische Ansatze*, Mentis, 2001, pp. 297–310.

[14] Ivlev, Y. V., Quasi-matrix logic, *Journal of Multi. Val. Logic and Soft Computing* 11(3-4):239–252, 2005.

[15] Karpenko, A. S., *Many-valued logic*, Moscow, 1997 (in Russian).

[16] Rescher, N., *Many-valued logic*, N.Y., 1969.