How Peircean was the "'Fregean' Revolution" in Logic?[1]

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abstract. The work in logic of Charles Peirce is surveyed in light of the characteristics enumerated by historian of logic J. van Heijenoort as defining the original innovations in logic of Frege and which together are said to be the basis of what has come to be called the "Fregean revolution" in logic and which are said to constitute the elements of Frege's Begriffsschrift of 1879 as the "founding" document of modern logic.

Keywords: history of logic, modern logic, algebraic logic, abstract algebraic logic; propositional logic; first-order logic; quantifier elimination, equational classes, relational systems

As editor of the very influential anthology *From Frege to Gödel* [104] (hereafter FFTG), historian of logic Jean van Heijenoort (1912–1986) did as much as anyone to canonize as historiographical truism the conception, initially propounded by Bertrand Russell (1872–1970), that modern logic began with the publication in 1879 of the *Begriffsschrift* [24] of Gottlob Frege (1848–1925), and thereby establishing Frege as the founder of modern logic. Van Heijenoort [104, p. vi] did this by relegating, as a minor sidelight in the history of logic, perhaps "interesting in itself" but of little historical impact, the tradition of algebraic logic of George Boole (1815–1864), Augustus De Morgan (1806–1871), Charles Sanders Peirce (1839–1914), William Stanley Jevons (1835–1882), John Venn (1834–1923), and Ernst Schröder (1841–1902).

The attitudes of Frege, as expressed, e.g., in his review of Schröder's *magnum opus* (see, e.g. [21]), and Edmund Husserl (1859–1938) (see, e.g. [43]) toward algebraic logic were more strongly negative than even Russell's or van Heijenoort's. We recall, for example,

[1] A complete and detailed account of the historical and technical background for this survey is available at: `http://www.cspeirce.com/menu/library/aboutcsp/anellis/csp-frege-revolu.pdf`.

the chastisement by Schröder's student Andreas Heinrich Voigt (1860–1940) [109] of Husserl's assertion in "Der Folgerungscalcul und die Inhaltslogik" [43, p. 171] that algebraic logic is not logic, and Frege's ire at Husserl for regarding Schröder, rather than Frege, as the first in Germany to work in symbolic logic. Not only that; Voigt pointed out that much of what Husserl claimed as original for his own work in logic was already to be found in Frege and Peirce.

In his review of the first volume of Schröder's *Vorlesungen über die Algebra der Logik* [90], Frege [29, p. 452] wrote that: "Alles dies ist sehr anschaulich, unbezweifelbar; nur schade: es ist unfruchtbar, und es ist keine Logik." [All this is very intuitive, undoubtedly; just a shame: it is unfruitful, and it is no logic.]

Russell was one of the most enthusiastic early supporters of Frege and contributed significantly to the conception of Frege as the originator of modern mathematical logic, although he never explicitly employed the specific term "Fregean revolution". In his recollections, he states that many of the ideas that he thought he himself originated, he later discovered had already been first formulated by Frege (see, e.g. [34, p. 245], for Russell's letter to Louis Couturat (1868–1914) of 25 June 1902), and some others were due to Giuseppe Peano (1858–1932) or the inspiration of Peano.

Russell's extant notes and unpublished writings demonstrate that significant parts of logic that he claimed to have been the first to discover were already present in the logical writings of Peirce and Schröder (see [1] and [3] for details)[2]. With regard to Russell's claim, to having invented the logic of relations, he was later obliged to admit (see [3, p. 281], quoting a letter to Couturat of 2 June 1903) that Peirce and Schröder had already "treated" of the subject, so that, in light of his own work, it was unnecessary to "go through" them.

We also find that Russell not only read Peirce's "On the Algebra of Logic" of 1880 [65] and "On the Algebra of Logic: A Contribution to the Philosophy of Notation" of 1885 [69] and the first volume of Schröder's *Vorlesungen über die Algebra der Logik* earlier than his

[2]Russell's library, including manuscripts and notes, are held at The Bertrand Russell Archives, The William Ready Division of Archives and Research Collections, Mills Memorial Library, McMaster University, Hamilton, Ontario, Canada.

statements suggest: there are extant notes for these dating from ca. 1900–1901 (see [1] and [3, p. 282]), and had known the work and many results even earlier, in the writing of his teacher Alfred North Whitehead (1861–192), as early as 1898, if not earlier, when reading the galley proofs of Whitehead's *Treatise of Universal Algebra* [110] of 1898, coming across references again in Peano, and was being warned by Couturat not to short-change the work of the algebraic logicians (see [3] for details).

What historiography of logic calls the "Fregean revolution" was articulated in detail by Jean van Heijenoort.

In FFTG [104], and which historiography of logic has for long taken as embracing all of the significant work in mathematical logic, van Heijenoort described Frege's *Begriffsschrift* of 1879 [24] as of singular significance for the history of logic, comparable, if at all, only with Aristotle's *Prior Analytics*, as opening "a great epoch in the history of logic..." [104, p. vi]. In his posthumously published "Historical Development of Modern Logic" [106], originally written in 1974, he makes the point more forcefully still of the singular and unmatched significance of Frege and his *Begriffsschrift* booklet of a mere 88 pages; he began this essay with the unequivocal and unconditional declaration [106, p. 242] that: "Modern logic began in 1879, the year in which Gottlob Frege (1848–1925) published his *Begriffsschrift*." Van Heijenoort goes on to explain [106, p. 242] that: "In less than ninety pages this booklet presented a number of discoveries that changed the face of logic. The central achievement of the work is the theory of quantification; but this could not be obtained till the traditional decomposition of the proposition into subject and predicate had been replaced by its analysis into function and argument(s). A preliminary accomplishment was the propositional calculus, with a truth-functional definition of the connectives, including the conditional. Of cardinal importance was the realization that, if circularity is to be avoided, logical derivations are to be *formal*, that is, have to proceed according to rules that are devoid of any intuitive logical force but simply refer to the typographical form of the expression; thus the notion of formal system made its appearance. The rules of quantification theory, as we know them today, were then introduced. The last part of the book belongs to the *foundations of mathematics*, rather than to logic, and presents a logical definition of the notion of mathematical sequences. Frege's contribution marks one of the sharpest breaks that ever occurred in the development of a science".

Frege's friend and University of Jena colleague Paul Ferdinand Linke (1876–1955) helped disseminate the concept of a Fregean revolution, writing, at a time when the ink was barely dry on the second edition of Whitehead and Russell's *Principia Mathematica* (1925–27) [112], when he wrote [51, pp. 226–227]: "...the great reformation in logic... originated in Germany at the beginning of the present century... was very closely connected, at least at the outset, with mathematical logic. For at bottom it was but a continuation of ideas first expressed by the Jena mathematician, Gottlob Frege. This prominent investigator has been acclaimed by Bertrand Russell to be the first thinker who correctly understood the nature of numbers. And thus Frege played an important role in... mathematical logic, among whose founders he must be counted".

We cannot help but notice a significant gap in the choices of material included in FFTG — all of the work of the algebraic logicians are absent, not just for Boole and De Morgan, whose work began to appear in the 1840s, and for their most influential and popular followers, Jevons and Venn, whose work appeared in the critical period of the 1870s up to 1900, and even for the work by Peirce and Schröder that appeared in the years which FFTG, a work purporting to completeness, and that despite the fact that FFTG includes work that refer back, often explicitly, to contributions in logic by Peirce and Schröder, even while Frege and his work remains virtually unmentioned in any of the other selections found in FFTG. The exclusion of Peirce and Schröder in particular from FFTG is difficult to understand if for no other reason than that their work is cited by many of the other authors whose work is included, and in particular is utilized by Leopold Löwenheim (1878–1957) and Thoralf Albert Skolem (1887–1963), whereas Frege's work is hardly cited at all in any of the other works included in FFTG; the most notable exceptions being the exchange between Russell and Frege concerning Russell's discovery of his paradox [104, pp. 124–128] and Russell's references to Frege in his paper of 1908 on theory of types ([86]; see [104, pp. 150–182]). The work of the algebraic logicians is excluded because, in van Heijenoort's estimation, and in that of the majority of historians and philosophers — almost all of whom have since at least the 1920s, accepted this judgment, that the work of the algebraic logicians falls outside of the Fregean tradition, and therefore does not belong to modern mathematical logic. Van Heijenoort makes the distinction as one primarily between algebraic logicians, most notably Boole, De Morgan, Peirce, and Schröder, and logicians who worked in quantification theory, first of all Frege, and with Russell as his most notable follower. For that, the logic that Frege created, as distinct from algebraic logic, was *mathematical* logic.

Hans Sluga [98], following van Heijenoort's distinction between followers of Boole and followers of Frege, labels the algebraic logicians "Boolean" and distinguishes them from the "Fregeans". The most important member of the Fregeans being Russell, the Booleans including not only of course Boole and De Morgan, but logicians such as Peirce and Schröder who combined, refined, and further developed the algebraic logic and logic of relations established by Boole and De Morgan.

Russell, in addition to the strong and well-known influence which Peano had on him, was a staunch advocate, and indeed one of the earliest promoters, of the conception of a "Fregean revolution" in logic, although he himself never explicitly employed the term itself. Nevertheless, we have such pronouncements, for example in his manuscript on "Recent Italian Work on the Foundations of Mathematics" of 1901 (see [87, pp. 350–362]) in which he contrasts the conception of the algebraic logicians with that of Hugh MacColl (1837–1909) and Frege, by writing that: "It has been one of the bad effects of the analogy with ordinary Algebra that most formal logicians (with the exception of Frege and Mr. MacColl) have shown more interest in logical equations than in implication". This view was echoed by van Heijenoort, whose chief complaint regarding the algebraic logicians was that they "tried to copy mathematics too closely, and often artificially" [104, p. vi].

In elaborating the distinguishing characteristics of mathematical logic and, equivalently, enumerating the innovations which Frege-allegedly-wrought to create mathematical logic, van Heijenoort (in "Logic as Calculus and Logic as Language" [105, p. 324]) listed:

1) a propositional calculus with a truth-functional definition of connectives, especially the conditional;

2) decomposition of propositions into function and argument instead of into subject and predicate;

3) a quantification theory, based on a system of axioms and inference rules; and

4) definitions of *infinite sequence* and *natural number* in terms of logical notions (i.e. the logicization of mathematics).

In addition, Frege, according to van Heijenoort and adherents of the historiographical conception of a "Fregean revolution":

5) presented and clarified the concept of *formal system*; and

6) made possible and gave a use of logic for philosophical investigations (especially for philosophy of language).

Moreover, in the undated, unpublished manuscript notes "On the Frege-Russell Definition of Number"[3], van Heijenoort claimed that Russell was the first to introduce a means for

7) separating singular propositions, such as "Socrates is mortal" from universal propositions such as "All Greeks are mortal"

among the "Fregeans". Yet, judging the "Fregean revolution" by the (seven) supposedly defining characteristics of modern mathematical logic, we should include Peirce as one of its foremost participants, if not one of its initiators and leaders. At the very least, we should count Peirce and Schröder among the "Fregean's rather than the 'Booleans' were they are ordinarily relegated and typically have been dismissed by such historians as van Heijenoort as largely, if not entirely, irrelevant to the history of modern mathematical logic, which is 'Fregean' ".

Donald Gillies [32] is perhaps the leading contemporary adherent and advocate of the conception of the "Fregean revolution", and has

[3]Held in Box 3.8/86-33/2 of Van Heijenoort Nachlaß: Papers, 1946–1983; Archives of American Mathematics, University Archives, Barker Texas History Center, University of Texas at Austin.

emphasized in particular the nature of the revolution a replacement of the ancient Aristotelian paradigm of logic by the Fregean paradigm. The centerpiece of this shift is the replacement of the subject–predicate syntax of Aristotelian propositions by the function–argument syntax offered by Frege (i.e. van Heijenoort's second criterion). The Booleans are numbered among the Aristotelians because they adhere to the subject–predicate syntax.

Whereas van Heijenoort and Willard Van Orman Quine (1908–2000) (see, e.g. [81, p. i]) stressed in particular the third of the defining characteristics of Fregean or modern mathematical logic, the development of a quantification theory, Gillies [32] argues in particular that Boole and the algebraic logicians belong to the Aristotelian paradigm, since, he explains, they understood themselves to be developing that part of Leibniz's project for establishing a *mathesis universalis* by devising an arithmeticization or algebraicization of Aristotle's categorical propositions and therefore of Aristotelian syllogistic logic, and therefore retaining, despite innovations in symbolic notation that they devised, the subject–predicate analysis of propositions.

What follows is a quick survey of Peirce's work in logic, devoting attention to Peirce's contributions to all seven of the characteristics that purportedly distinguish the Fregean from the Aristotelian or Boolean paradigms. While concentrating somewhat on the first, where new evidence displaces Jan Łukasiewicz (1878–1956), Emil Leon Post (1897–1954), and Ludwig Wittgenstein (1889–1951) as the originators of truth tables, and on the third, which most defenders of the conception count as the single most crucial of those defining characteristics. The replacement of the subject–predicate syntax with the function–argument syntax is ordinarily accounted of supreme importance, in particular by those who argue that the algebraic logic of the "Booleans" is just the symbolization, in algebraic guise, of Aristotelian logic. But the question of the nature of the quantification theory of Peirce, Oscar Howard Mitchell (1851–1889), and Schroeder as compared with that of Frege and Russell is tied up with the ways in which quantification is handled.

The details of the comparison and the mutual translatability of the two systems is better left for another discussion. Suffice it here to say that Norbert Wiener (1894–1964) dealt with the technicalities in detail in his doctoral thesis for Harvard University of 1913, *A Comparison Between the Treatment of the Algebra of Relatives by Schröder and that by Whitehead and Russell* [113], and concluded that there is nothing that can be said in the *Principia Mathematica* [111] of Whitehead and Russell that cannot, with equal facility, be said in the Peirce–Schröder calculus, as presented in Schröder's *Algebra der Logik* [90][4]. After studying logic with Josiah Royce (1855–1916) and Peirce's correspondent Edward V. Huntington (1874–1952), Wiener went on for post-graduate study at Cambridge University with Whitehead, and debated with Russell concerning the results of his doctoral dissertation. Russell claimed in reply that Wiener considered only "the more conventional parts of *Principia Mathematica*" (see [33, p. 130]).

With that in mind, I want to focus attention on the question of quantification theory without ignoring the other points.

1 Peirce's propositional calculus with a truth-functional definition of connectives, especially the conditional

Consider the following formulas:

Peano–Russell: $[(\sim c \supset a)\supset(\sim a \supset c)]\supset\{(\sim c \supset a)\supset[(c \supset a) \supset a]\}$

Peirce: $[(\bar{c} \prec a) \prec (\bar{a} \prec c)] \prec \{(\bar{c} \prec a) \prec [(c \prec a) \prec a]\}$

Schröder: $[(c' \mathrel{\text{€}} a) \mathrel{\text{€}} (a' \mathrel{\text{€}} c)] \mathrel{\text{€}} \{(c' \mathrel{\text{€}} a) \mathrel{\text{€}} [(c \mathrel{\text{€}} a) \mathrel{\text{€}} a]\}$

Clearly, for propositional logic, the differences are entirely and solely notational[5].

In the manuscript "On the Algebraic Principles of Formal Logic"[6], written in the autumn of 1879 — the very year in which Frege's *Begriffsschrift* appeared, Peirce (see [73, p. 23]) explicitly identified his "claw" or "hook" of illation ($\prec$) as the "copula of inclusion" and defined material implication or logical inference, *illation*, as "1st, $A \prec A$, whatever A may be. 2nd, If $A \prec B$, and $B \prec C$, then $A \prec C$." From there, he immediately connected his definition with

[4][33] is an expository survey of Wiener's thesis. [6, pp. 429–444] reproduces the introduction and concluding chapter of [113].

[5]See, e.g. [22] on Peirce's propositional logic.

[6]Peirce's Nachlaß was originally located in Harvard University's Widener Library and is now located in Harvard's Houghton Library, with copies of all materials located in the Max H. Fisch Library at the Institute for American Thought, Indiana University–Purdue University at Indianapolis [IUPUI].

truth-functional logic, by asserting that: "This definition is sufficient for the purposes of formal logic, although it does not distinguish between the relation of inclusion and its converse. Were it desirable thus to distinguish, it would be sufficient to add the real truth or falsity of $A \prec B$, supposing the existence of A". The following year, Peirce continued along this route: in "The Algebra of Logic" of 1880 [65, p. 8, 21], he wrote that $A \prec B$ is explicitly defined as "A implies B", and $A \overline{\prec} B$ defines "A does not imply B".

Moreover, we are able to distinguish universal and particular propositions, affirmative and negative, according to the following scheme:

A.	$a \prec b$	All A are B	(universal affirmative)
E.	$a \prec \bar{b}$	No A are B	(universal negative)
I.	$\breve{a} \prec b$	Some A are B	(particular affirmative)
O.	$\breve{a} \prec b$	Som A are not B	(particular negative)

And in 1885 in "On the algebra of logic: a contribution to the philosophy of notation" [69, p. 184, 186–187], Peirce sought to redefine categoricals as hypotheticals and presented a propositional logic.

In the manuscript fragment "Algebra of Logic (Second Paper)" written in the summer of 1884, Peirce reiterated his definition of 1880, and explained in greater detail there that: "In order to say 'If it is a it is b', let us write $a \prec b$. The formulae relating to the symbol $\prec$ constitute what I have called the algebra of the copula... The proposition $a \prec b$ is to be understood as true if either a is false or b is true, and is only false if a is true while b is false".

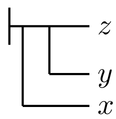
It was at this stage that Peirce undertook the truth-functional analysis of propositions and of proofs, and also introduced specific truth-functional considerations, saying that, for $\mathbf{v}$ the symbol for "true" and $\mathbf{f}$ the symbol for "false", the propositions $\mathbf{f} \prec a$ and $a \prec \mathbf{v}$ are true, and either one or the other of $\mathbf{v} \prec a$ or $a \prec \mathbf{f}$ are true, depending upon the truth or falsity of a, and going on to further analyze the truth-functional properties of the "claw"[7].

[7]For the historical background for Peirce's work in truth-functional logic and his development of truth tables, along with an account of his own work in truth-functional logic and truth tables for triadic logic, see [3]. For further details, and an account of Peirce's truth tables for bivalent logic of 1883–93, see [4].

In Peirce's conception, as found in his "Description of a notation for the logic of relatives..." of 1870 [64], the Aristotelian syllogism becomes a hypothetical proposition, with material implication as its main connective; he writes *Barbara* as

$$\begin{array}{rl} \text{If} & x \mathrel{-\!\!<} y, \\ \text{and} & y \mathrel{-\!\!<} z \\ \text{then} & x \mathrel{-\!\!<} z. \end{array}$$

In Frege's *Begriffsschrift* notation [24, §6] this same argument would be rendered as



which, in the familiar Peano-Russell notation, is just $[(x \supset y) \bullet (y \supset z)] \supset (x \supset z)$. Ironically, Schröder, even complained about what he took to be Peirce's (and MacColl's) efforts to base logic on the propositional calculus, which he called the "MacColl-Peircean propositional logic" (see [90, I, pp. 89-592] and especially [90, II, p. 276]).

John Shosky [95] distinguished between the truth table technique, what we typically call *truth-functional analysis*, from the truth table device, the arrangement of truth-functional analysis in matrix form, what we typically call the *truth table*. In opposition to the canonical view that the earliest identifiable truth tables were presented, nearly simultaneously, between 1920 and 1922 by Łukasiewicz [53], Post [76] and [77], and Wittgenstein [114], Shosky provided an example of truth tables discovered on the verso of a typescript by Russell dating from 1912. He neglects the evidence that Peirce had devised truth tables for a trivalent logic as early as 1902–09 and had worked out a truth table for the sixteen binary propositional connectives, the latter based upon the work of Christine Ladd [46, esp. p. 62], which in turn was based upon the work of Jevons [44, p. 135] (see [71, 4.262]; see also [49], [23], [15], [103], [115], as well as [49], [4], and [5]). Moreover, while carrying out his work in 1883–84 on what was to planned as the second half of the article of 1880 "On the Algebra of Logic" for the *American Journal of Mathematics* on the algebra of relations, Peirce produced a manuscript "On the Algebra of Logic" and the accompanying supplement, in which we find what unequivocally would today be labeled as an indirect or abbreviated truth table for the formula $\{\overline{((a -\!\!< b)} -\!\!< c) -\!\!< d\} -\!\!< e$, as follows:

$$
\begin{array}{c}
\{\overline{((a -\!\!< b)} -\!\!< c) -\!\!< d\} -\!\!< e \\
\begin{array}{cccccc}
\mathbf{f} & \mathbf{f} & \mathbf{f} & \mathbf{f} & -\!\!< & \mathbf{f} \\
\mathbf{f} & \mathbf{v} & & \mathbf{v} & & \mathbf{f} \\
- & - & - & - & \mathbf{v} &
\end{array}
\end{array}
$$

(see [4]). The whole of the undated eighteen-page manuscript "Logic of Relatives", also identified as composed *circa* 1883–84, is devoted to a truth-functional analysis of the conditional, which includes the equivalent, in list form, of the truth-table for $x -\!\!< y$, as follows:

$$
\begin{array}{cccc}
 & \multicolumn{2}{c}{x -\!\!< y} & \\
 & \text{is true} & \text{is false} & \\
 & \text{when} & \text{when} & \\
x = \mathbf{f} & y = \mathbf{f} & x = \mathbf{v} & y = \mathbf{f} \\
x = \mathbf{f} & y = \mathbf{v} & & \\
x = \mathbf{v} & y = \mathbf{v} & &
\end{array}
$$

Peirce also wrote there that: "It is plain that $x -\!\!< y -\!\!< z$ is false only if $x = \mathbf{v}$, $(y -\!\!< z) = \mathbf{f}$, that is only if $x = \mathbf{v}, y = \mathbf{v}, z = \mathbf{f}\ldots$"

Finally, in the undated manuscript "An Outline Sketch of Synechistic Philosophy" identified as composed in 1893, we have an unmistakable example of a truth-table matrix for a proposition and its negation, as follows:

	t	f
t	t	f
f	t	t

which is clearly and unmistakably equivalent to the truth-table matrix for $x -\!\!< y$ in the contemporary configuration, expressing the same values as we note in Peirce's list in the 1883–84 manuscript "Logic of Relatives". That the multiplication matrices are the most probable inspiration for Peirce's truth-table matrix is that it appears alongside matrices for a multiplicative two-term expression of linear algebra for $\{i, j\}$ and $\{i, i - j\}$. Indeed, it is virtually the same table, and in roughly — i.e., apart from inverting the location within the respective tables for antecedent and consequent — the same configuration as that found in the notes, taken in April 1914 by Thomas Stearns Eliot (1888–1965) in Russell's Harvard University logic course (as reproduced at [95, p. 23]), where we have:

$$p \vee q \quad q\left\{\begin{array}{c|c|c} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{p}} \\ & \mathrm{T} & \mathrm{F} \\ \hline \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \hline \mathrm{F} & \mathrm{T} & \mathrm{F} \end{array}\right. \qquad p \supset q \quad q\left\{\begin{array}{c|c|c} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{p}} \\ & \mathrm{T} & \mathrm{F} \\ \hline \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \hline \mathrm{F} & \mathrm{F} & \mathrm{T} \end{array}\right. \qquad {\sim}p \vee {\sim}q \quad q\left\{\begin{array}{c|c|c} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{p}} \\ & \mathrm{T} & \mathrm{F} \\ \hline \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \hline \mathrm{F} & \mathrm{T} & \mathrm{F} \end{array}\right.$$

The first published instance by Peirce of a truth-functional analysis which satisfies the conditions for truth tables, but not as yet constructed in tabular form, is in Peirce's 1885 paper "On the Algebra of Logic: A Contribution to the Philosophy of Notation" [69, p. 189–190] in which he gave a proof, using the truth-table method of what has come to be known as *Peirce's Law*: $((A \to B) \to A) \to A$, his "fifth icon", whose validity he tested using truth-functional analysis. In an untitled paper written in 1902 as subsequently published posthumously [71, 260–262][8], Peirce displayed the following table for three terms, x, y, z, writing **v** for *true* and **f** for *false*:

x	y	z
v	**v**	**v**
v	**f**	**f**
f	**v**	**f**
f	**f**	**v**

where z is the term derived from an [undefined] logical operation on the terms x and y. In February 1909, while working on his trivalent logic, Peirce applied the tablular method to various connectives, for example negation of x, as $\overline{x}$, in his *Logic Notebook 1865–1909* (see [23]), and with **V**, **F**, and **L** are the truth-values true, false, and indeterminate or unknown respectively, which he called "limit"[9].

[8]Now identified as "The Simplest Mathematics"; January 1902 (Chapter III. The Simplest Mathematics (Logic III)). For the truth table matrix, see [71, 4.262].

[9]Under the title "On Triadic Logic", the relevant fragment, dated 23 February 1909, was published in [36, p. 217–224].

Russell's rendition of Wittgenstein's tabular definition of negation, as written out on the verso of a page from Russell's transcript notes, using 'W' (wahr) and 'F' (falsch) (see [95, p. 20]), where the negation of p is written out by Wittgenstein as $p \between q$, with Russell adding '$=\sim p$'', to yield: $p \between q =\sim p$ is

p	q
W	W
W	F
F	W
F	F

Within the same time frame as the work on truth tables of Post, Łukasiewicz, and Wittgenstein, is the work of Ivan Ivanovich Zhegalkin (1896–1947), who, independently provided a Boolean-valued truth-functional analysis of propositions of propositional logic [117] and its extension to first-order logic [118], undertaking to apply truth tables to the formulas of propositional calculus and first-order predicate calculus. He employed a technique resembling those employed by Peirce-Mitchell-Schröder, Löwenheim, Skolem, and Jacques Herbrand (1908–1931) to write out an expansion of logical polynomials and assigning them Boolean values[10].

2 Decomposition of propositions into function and argument instead of into subject and predicate

In the opening sentence of his *Methods of Logic* [81, p. i], clearly referring to the year that Frege's *Begriffsschrift* was published, Quine wrote: "Logic is an old subject, and since 1879 it has been a great one". J. Brent Crouch [18, p. 155], quoting Quine, takes this as evidence that historiography continues to hold Frege's work as seminal and the origin of modern mathematical logic, and appears in the main to concur, saying that Frege's *Begriffsschrift* is "one of the first published accounts of a logical system or calculus with quantification and a function-argument analysis of propositions. There can be no doubt as to the importance of these introductions,

[10] See Part 2, "3. Peirce's quantification theory, based on a system of axioms and inference rules" for equations of the logic of relatives as logical polynomials and the Peirce-Schröder method of expansion of quantified formulas as logical sums and products.

and, indeed, Frege's orientation and advances, if not his particular system, have proven to be highly significant for much of mathematical logic and research pertaining to the foundations of mathematics". This ignores a considerably important aspect of the history of logic, and more particularly much of the motivation which the Booleans had in developing a "symbolical algebra".

The "Booleans" were well acquainted, from the 1820s onward, with the most recent French work in function theory of their day, and although they did not explicitly employ a function-theoretic syntax in their analysis of propositions, they adopted the French algebraic approach, favored by Joseph-Louis Lagrange (1736–1813), Adrien-Marie Legendre (1752–1833), and Augustin-Louis Cauchy (1789–1857), to functions over the function-argument syntax which Frege adapted from the analysis, including in particular as found in the work of his teacher Karl Weierstrass (1815–1897). So there was some justification in the assertion by Russell that the algebraic logicians were more concerned with logical equations than with implication[11].

We see this in the way that the Peirceans approached indexed logical polynomials. It it easier to understand the full implications when examined from the perspective of quantification theory. But, as a preliminary, we can consider Peirce's logic of relations and how to interpret these function-theoretically.

If an early example is wanted, consider, e.g., Boole's definition in *An Investigation of the Laws of Thought* [12, p. 71]: "Any algebraic expression involving the symbol x is termed a function of x, and may be represented under the abbreviated form $f(x)$," following which binary functions and n-ary functions are allowed, along with details for dealing with these as elements of logical equations in a Boolean-valued universe.

We may summarize the crucial distinctions by describing the core of Aristotle's formal logic as a syllogistic logic, or logic of terms, and the propositions and syllogisms of logic having a subject-predicate syntax, entirely linguistic, the principle connective for which, the copula, is the copula of existence, which is metaphysically based

[11] On the connections between the work in the algebra of functions, or "operational calculus" and the development of algebraic logic, in particular by De Morgan and Boole, see, e.g. [47], [48], [79], [80], and [59]. For Boole's work in particular on the role of functions in his algebraic logic, see, e.g. [84].

and concerns the inherence of a property, whose reference is the predicate, in a subject; Boole's formal logic as a logic of classes, the terms of which represent classes, and the copula being the copula of class inclusion, expressed algebraically; and De Morgan's formal logic being a logic of relations whose terms are relata, the copula for which is a relation, expressed algebraically. It is possible then to say that Peirce in his development dealt with each of these logics, Aristotle's, Boole's, and De Morgan's, in turn, and arrived at a formal logic which combined, and then went beyond, each of these, by allowing his copula of illation to hold, depending upon context, for terms of syllogisms, classes, and propositions, and expanding these to develop a quantification theory as well, in his logic of relatives.

Gilbert Ryle (1900–1976) although admittedly acknowledging that the idea of *relation* and the resulting relational inferences were "made respectable" by De Morgan nevertheless attributed to Russell, in *The Principles of Mathematics* [85] — rather than to Peirce — their codification and to Russell — rather than to Peirce and Schröder — their acceptance, again by Russell in the *Principles*. Ryle [88, p. 9–10] wrote: "The potentialities of the xRy relational pattern, as against the overworked s–p pattern, were soon highly esteemed by philosophers, who hoped by means of it to order all sorts of recalitrances in the notions of knowing, believing. . . ".

Mitchell [55] defined indexed logical polynomials, such as '$l_{i,j}$', as functions of a class of terms, in which for the logical polynomial F as a function of a class of terms $a, b, \ldots,$ of the universe of discourse U, $F1$ is defined as "All U is F" and Fu is defined as "Some U is F", and Peirce defined identity in second-order logic on the basis of Leibniz's Identity of Indiscernibles, as $l_{i,j}$, meaning that every predicate is true/false of both i, j. What Mitchell produces is a refinement of the notation that Peirce himself had devised for his algebra of relatives from 1867 forward, enabling the distinction between the terms of the polynomials by indexing of terms, and adding the index of the quantifiers ranging over the terms of the polynomials. Mitchell's improvements were immediately adopted by Peirce [68] and enabled Peirce to develop, as we shall see, a first-order quantification theory fully as expressive as Frege's.

The necessary apparatus to translate between relational expressions and functional expresses is provided, in contemporary terms, by Ramsey's Maxim (see, e.g. [111, I, p. 27]). Suppose that we have a binary relation aRb. This is logically equivalent to the function-theoretic expression $R(a, b)$, where R is a binary function taking a and b as its arguments. A function is a relation, but a special kind of relation, then, which associates one element of the *domain* (the universe of objects or terms comprising the arguments of the function) to precisely one element of the *range* (or *codomain*, the universe of objects or terms comprising the values of the function). So, clearly, Pierce's logical polynomials expressing relations can be rewritten in function-theoretical terms.

This takes us to the next point: that among Frege's creations that characterize what is different about the mathematical logic created by Frege and helps define the "Fregean revolution", *viz.*, a quantification theory based on a system of axioms and inference rules.

3 Peirce's quantification theory based on a system of axioms and inference rules

Despite numerous historical evidences to the contrary, as has been suggested as long ago as the 1950s, e.g. by — in chronological order — George D. W. Berry [11], Richard Beatty [10], and the late Richard Milton Martin (1916–1985) [54], we still find, even in the most recent issue of the Peirce *Transactions*, repetition by J. Brent Crouch [18] of the old assertion by Quine from his *Methods of Logic* [81, p. i]. Crouch [18, p. 155] thus writes: "In the opening sentence of his *Methods of Logic*, W. V. O. Quine writes, 'Logic is an old subject, and since 1879 it has been a great one'. Quine is referring to the year in which Gottlob Frege presented his *Begriffsschrift*, or 'concept–script', one of the first published accounts of a logical system or calculus with quantification and a function–argument analysis of propositions. There can be no doubt as to the importance of these introductions, and, indeed, Frege's orientation and advances, if not his particular system, have proven to be highly significant for much of mathematical logic and research pertaining to the foundations of mathematics". And this despite the fact that Quine himself eventually repudiated this assertion, long before it came to the attention of Crouch. Quine himself, that is, ultimately acknowledged in 1985 [82] and again in 1995 [83] that Peirce had developed a quantification theory just a few years after Frege. More accurately, Quine began developing a quantification theory more than a decade prior to the publication of Frege's *Begriffsschrift* of 1879 [24], but admittedly did not have a fully developed quantification theory until 1885, six years after the appearance of the *Begriffsschrift*.

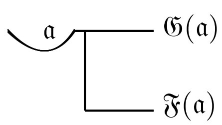
Peirce's efforts to develop what was to become his first-order quantification theory in his 1883 "The Logic of Relatives" [68] begun at least as early as 1867, in his "On an Improvement in Boole's Calculus of Logic" [63] and further enhanced by the notational innovations by O. H. Mitchell in his "On a New Algebra of Logic" of 1883 [55], and more fully articulated and perfected two years later in Peirce's "On the Algebra of Logic: A Contribution to the Philosophy of Notation" [69] in which we have not only a first-order quantification theory but also a second-order quantification theory, the latter admittedly not yet as well developed as the first-order theory. But even more importantly that Pierce's system dominated logic in the final decades of the nineteenth century and first two decades of the twentieth largely through the efforts of Schröder, in particular in his *magnum opus*, *Vorlesungen über die Algebra der Logik* [90], whereas Frege's work exerted scant influence[12], and that largely negative, until brought to the attention of the wider community by Russell, beginning with his *Principles of Mathematics* of 1903 [85], largely through the introduction of and efforts to circumvent or solve the Russell paradox. Thus, by 1885, Peirce had not only a fully developed first-order theory, which he called the *icon*

[12][14] collects and provides English translations of most of the reviews of the *Begriffsschrift* that appeared immediately following its publication and gives the canonical view of the reception of Frege's *Begriffsschrift* [14, p. 15–20]. Most of those reviews, like Venn's [107], were only a few pages long, if that, and emphasized the "cumbrousness" of the notation and lack of originality of the contents. The most extensive and what was that of Schröder [89], which remarks upon the lack of originality and undertakes a detailed discussion of the contents as compared with his own work and the work of Peirce, criticizing in particular Frege's failure to familiarize himself with the work of the algebraic logicians. Schröder's review sparked a literary battle between himself and Frege and their respective defenders. [99] advances the conception that Frege's reputation was established by Russell, while [108] argues that Frege's *Begriffsschrift* received a respectable amount of attention after its appearance in consideration of the fact of its authorship by an investigator making his first entry into the field.

of the second intention, but a good beginning at a second-order theory, as found in his "On the Algebra of Logic: A Contribution to the Philosophy of Notation" [69].

The final version of Peirce's first-order theory uses indices for enumerating and distinguishing the objects considered in the Boolean part of an equation as well as indices for quantifiers, a concept taken from Mitchell. Peirce denoted the existential and universal quantifiers by 'Σ_i' and 'Π_i' respectively, as logical sums and products, and individual variables $i, j, \ldots$, are assigned to both quantifiers and predicates, that is, to both quantifiers and to the distinct terms of the logical polynomials of the Boolean part of the equation. He then wrote '$l_{i,j}$' for 'i is the lover of j'. Then "Everybody loves somebody" is written in Peirce's quantified logic of relatives as $\Pi_i \Sigma_j l_{i,j}$ i.e. as "Everybody is the lover of somebody". In Peirce's own exact expression, as found in his "On the Logic of Relatives" [47, p. 200], we have "$\Pi_i \Sigma_j l_{i,j} > 0$ means that everything is a lover of something". That is, Peirce defined the existential and universal quantifiers by 'Σ_i' and 'Π_i' respectively, as logical sums and products, e.g., $\Sigma_i x_i = x_i + x_j + x_k \ldots$, and $\Pi_i x_i = x_i \exists x_j \exists x_k$, and individual variables $i, j, k, \ldots$, are assigned both to quantifiers and predicates. In the Peano–Russell notation these are of course $(\exists x)F(x) = F(x_i) \vee F(x_j) \vee F(x_k)$ and are $(\forall x)F(x) = F(x_i) \bullet F(x_j) \bullet F(x_k)$ respectively.

The difference between the Peirce–Mitchell–Schröder formulation, then, of quantified propositions is purely cosmetic, and both are significantly notationally simpler than Frege's. Frege's rendition of the proposition "For all x, if x is F, then x is G", i.e. $(\forall x)[F(x) \supset G(x)]$, for example, is



Not only that; recently, Calixto Badesa [8], [9] and Geraldine Brady [13] traced the details of the development of the origins of the special branches of modern mathematical logic known as *model theory*, which is concerned with the properties of consistency, completeness, and independence of mathematical theories including of course the various logical systems, and *proof theory*, concerned with studying the soundness of proofs within a mathematical or logical system. This route runs from Peirce and his student Mitchell, through Schröder to Löwenheim, Skolem, and — I would add — Herbrand. It was based upon the Peirce–Mitchell technique for elimination of quantifiers by quantifier expansion that the Löwenheim-Skolem Theorem [LST] allows logicians to determine the validity within a theory of the formulas of the theory and is in turn the basis for Herbrand's Fundamental Theorem [FT] which can best be understood as a strong version of LST[13].

Model theory, and especially LST, developed in large measure from Löwenheim's 1915 "Über Möglichkeiten im Relativkalkul" [52], Skolem's 1920 "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze. . . " [96] and Herbrand's 1930 doctoral thesis *Recherches sur la théorie des démonstration* [40], especially its fifth chapter, using Peirce and Schröder's techniques. Proof theory and especially FT arose in the the work of Herbrand [40] studying the work of Hilbert, using Peirce and Schröder's techniques and starting from the work of Löwenheim and Skolem, and most especially of Lowenheim. Meanwhile Alfred Tarski (1902—1983) inspired by the work of Peirce and Schröder, and Tarski's students, advanced algebraic logic, beginning in 1941 (see [101]; see also [3], [4], [7]) giving us Tarski's and Steven Givant's work (in particular in their *Formalization of Set Theory without Variables* [102] on Peirce's Fundamental Theorem (see [3]). Tarski's doctoral thesis (see [100]) displayed a keen awareness of Schröder's work; and both he [100] and Cooper Harold Langford (1895—1964) (see [50]) used the Peirce-Schröder quantifier elimination method as found in the work of Löwenheim and Skolem, although Langford gave no indication that he was aware of the work of either Löwenheim or Skolem (see [56, p. 248]). Tarski [101, p. 73–74] is very explicit in clearly expressing the influence and inspiration of Schröder, and especially of Peirce, as the origin of his own work.

The original version of what came to be known as LST as stated by Löwenheim [52, p. 450, §2, p. Satz 2] is simply that: *If a well-formed formula of first-order predicate logic is satisfiable, then it is* $\aleph_0$*-satisfiable.* Not only that: in the manuscript "The Logic of Relatives: Qualitative and Quantitative" of 1886 Peirce himself is

[13] [2] includes a discussion of the relation of FT to LST.

making use of what is essentially a finite version of LST (see [74, p. 374] and the associated note at [74, p. 464, n. 374.31-36]), that

If F is satisfiable in every domain, then F is $\aleph_0$-satisfiable;
that is:
If F is n-satisfiable, then F is $(n+1)$-satisfiable

and, indeed, his proof was in all respects similar to that which later appeared in Löwenheim's 1915 paper [52], where, for any $\kappa < \lambda$, a product vanishes (i.e. is satisfiable), and its κ^{th} term vanishes. In its most modern and strict form the LST says that: *For a κ-ary universe, a well-formed formula F is $\aleph_0$-satisfiable if it is κ-valid for every finite κ, provided there is no finite domain in which it is invalid.*

Herbrand's FT was developed in order to answer the question: *what finite sense can generally be ascribed to the truth property of a formula with quantifiers, particularly the existential quantifier, in an infinite universe?* The modern statement of FT is: *For some formula F of classical quantification theory, an infinite sequence of quantifier-free formulas $F_1, F_2, \ldots,$ can be effectively generated, for F provable in (any standard) quantification theory, if and only if there exists a κ such that F is (sententially) valid; and a proof of F can be obtained from F_κ.*

For both LST and FT the elimination of quantifiers carried out as expansion in terms of logical sums and products, as defined by Mitchell-Peirce-Schröder, is an essential tool. Moreover, it was precisely the Mitchell–Peirce–Schröder definition, in particular as articulated by Schröder in his *Algebra der Logik*, that provided this tool for Löwenheim, Skolem, and Herbrand.

4 Peirce's definition of infinite sequence and natural number in terms of logical notions (i.e. the logicization of mathematics)

Frege developed his theory of sequences defined in terms of logical notions in the third and final part of the *Begriffsschrift* [24, th. III, §§23–31], giving us first the ancestral relation and then the proper ancestral, the latter required in order to guarantee that the sequences arrived at are well-ordered. With the ancestral proper he is finally able to define mathematical induction as well.

In "On the Logic of Number" of 1881 [66] Peirce set forth an axiomatization of number theory, starting from his definition of *finite set*, to obtain natural numbers. Given a set N and R a relation on N, with 1 an element of N; with definitions of *minimum*, *maximum*, and *predecessor* with respect to R and N given, Peirce's axioms in modern terminology are:

1. N is partially ordered by R.
2. N is connected by R.
3. N is closed with respect to predecessors.
4. 1 is the minimum element of N; N has no maximum.
5. Mathematical induction holds for N.

It is in this context important to consider Sluga's testimony [97, p. 96–180] that it took Frege five years beyond the completion of date 18 December 1897 for the *Begriffschrift* to provide the promised elucidation of the concept of number following his recognition that there are logical objects and realizing that he had not successfully incorporated that recognition into the *Begriffsschrift*. Certainly, if Pierce in 1881 had not yet developed a complete and coherent logical theory of number, neither, then, had Frege before 1884 in *Die Grundlagen der Arithmetik* [28].

The only significant differences between the axiomatization of number theory by Richard Dedekind (1831—1914) in *Was sind und was sollen die Zahlen* [20] and Pierce's was that Dedekind started from *infinite sets* rather than finite sets in defining natural numbers, and that Dedekind is explicitly and specifically concerned with the real number continuum, that is, with infinite sets. This is because Peirce rejected the real continuum in favor of Leibnizian infinitesimals. Moreover, Peirce rejected transfinite sets maintaining the position that Cantor and Dedekind were unable to logically support the construction of the actual infinite, and that only the potential infinite could be established logically[14]. Nevertheless, Dedekind's set

[14]For discussions of Peirce's criticisms of Cantor's and Dedekind's set theory see e.g. [19] and [57].

theory and Peirce's and consequently their respective axiomatizations of number theory are equivalent. This equivalence of Peirce's axiomatization of natural numbers to that of Dedekind (as well as that of Giuseppe Peano's 1889 *Arithmetices principia* [60]) is demonstrated by Paul Bartram Shields (in [93] and [94]). The similarities between Peirce's axiom system and Dedekind's led Peirce to accuse Dedekind of plagiarizing his "Logic of Number". Francesco Gana [31] examined Peirce' claim against Dedekind and concluded that it was unjustified, that Dedekind was unfamiliar with Peirce's work.

Peirce did not turn his attention specifically and explicitly to infinite sets until engaging and studying the work of Dedekind and Georg Cantor (1845—1918), especially Cantor, and did not publish any of his further work in detail, although he did offer some hints in publications such as his "The Regenerated Logic" of 1896 [70][15].

The technical centerpiece of Dedekind's mathematical work was in number theory, especially algebraic number theory. His primary motivation was to provide a foundation for mathematics and in particular to find a rigorous definition of real numbers and of the real-number continuum upon which to establish mathematical analysis in the style of Karl Weierstrass. This means that he sought to axiomatize the theory of numbers based upon that rigorous definition of the real numbers and the construction of the real number system and the continuum which could be employed in defining the theory of limits of a function for use in the differential and integral calculus, real analysis, and related areas of function theory. His concern, in short, was with the rigorization and arithmetization of analysis.

For Peirce, on the other hand, the object behind his axiomatization of the system of natural numbers was stated in "On the Logic of Number" [66, p. 85] as establishing that "elementary propositions concerning number... are rendered [true] by the usual demonstrations". He therefore undertook "to show that they are strictly syllogistic consequences from a few primary propositions", and he asserted "the logical origin of these latter, which I here regard as definitions", but for the time being takes as given.

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[15] Some of Peirce's published writings and manuscripts relating to mathematics, including set theory, have recently appeared in [75].

In short. Peirce here wants to establish that the system of natural numbers can be developed axiomatically by deductive methods (i.e. "syllogistically") applying his logic of relations and that the system of natural numbers can be constructed by this means on the basis of some logical definitions[16].

Whether this is tantamount, from the philosopher's standpoint, to the logicism of Frege has been the subject of considerable debate and likewise depends upon whether one distinguishes, e.g., between logicism and "protologicism"[17].

5 Peirce's presentation and clarification of the concept of formal system

I would suggest that, even if Peirce nowhere formally and explicitly set forth his conception of a formal system, it is present and implicit in much of his work, in "On the Logic of Number" [66] for example, in the explication of the purpose of his project of deducing, in a logically coherent and explicit manner, and in strict accordance with deductive inference rules on the basis of a few essential and carefully chosen and well-defined "primary propositions" — definitions, and the propositions requisite for deriving and expressing the elementary propositions — axioms — of mathematics concerning numbers.

We consider the assertion by Geraldine Brady, who wrote in *From Peirce to Skolem* [13, p. 14] of Peirce's "failure to provide a formal system of logic, in the sense of Frege's. The motivation to create a formal system is lacking in Peirce...", and he thus "made no attempt at an all-encompassing formal system". We are constrained to admit that there is in Peirce no one set of axioms by which to derive all of logic, still less, all of mathematics. Rather, what we have is an on-going experiment in developing the basis of a logic that answers to specific purposes and has as its ultimate goal the creation of a calculus that serves as a tool for the wider conception

[16]We understand, here, however, that by the time he wrote those lines, Peirce had already translated syllogisms as implications within his algebraic logic (see "Peirce's propositional calculus with a truth-functional definition of connectives, especially the conditional").

[17]There is a vast literature on the question of whether or not Peirce was or was not a logicist, and, if so, to what extent; part of a lengthy and continuing debate, [35], [41] [58], and [21] are among a very small sampling of the more recent entries in this discussion.

of logic as a theory of signs. Rephrased, Peirce did not, either in one complete and coherent work or even over time, produce a single, all-encompassing formal system; rather, he produced a series of formal systems, often informally presented. Moreover, these systems, much like the systems of his predecessors and colleagues, most notably De Morgan, Boole, and Schröder, worked not with a universal universe of discourse, or universal domain, such as Frege's *Universum*, but with specific universes of discourse. Any one of his formal systems, that is, applied to what Schröder termed a *Gebiet*. In other words, Peirce presented formal systems each one of which was, in Schröder's terminology, a *Gebietkalkul*. (This is the basis of the discussion between Peano and Schröder of the comparative values of Peano's pasigraphy and Peirce's logical system[18].)

6 Peirce's logic and semiotics, making possible, and giving, a use of logic for philosophical investigations (especially for philosophy of language)

Van Heijenoort's understanding of Frege's conception of application of his logical theory for philosophical investigations and in particular for philosophy of language can be seen as two-fold, although van Heijenoort in particular instances envisioned it in terms of analytic philosophy. On the one hand, Frege's logicist program was understood as the centerpiece, and concerned the articulation of sciences, mathematics included, developed within the structure of the logical theory; on the other hand, it is understood more broadly as developing the logical theory as a universal language.

Distinguishing *logic as calculus* and *logic as language* van Heijenoort [104, p. 1–2] (see also [105]), taking his cue directly from Frege (see [24, p. XI]) understood the "Booleans" or algebraic logicians as concerned to treat logic as a mere calculus (see [25], [26], [27], [29]), whereas Frege and the "Fregeans" see their logic to be both a calculus and a language, but first and foremost as a language. It is in this regard that Frege (in [29]) criticized Schröder, although he had the entire algebraic tradition in mind, from Boole to Schröder (see

[18] See Schröder [91] and [92], responding to specific Peano's claim, at [61, p. 52], and in general to the work of Peano and his school in their publications in the *Rivista di matematiche* and the *Formulario*; see also [62]).

[25], [26] [27], [29]). This was in response to Schröder's assertion, in his review of 1880 of Frege's *Begriffsschrift*, that Frege's system "does not differ essentially from Boole's formula language", adding: "With regard to its major content the *Begriffsschrift* could actually be considered a *transcription* of the Boolean formula language" [89, p. 83][19].

As early as 1865 Peirce defined logic as the science of the conditions which enable symbols in general to refer to objects[20]. For Peirce (as expressed in "The Nature of Mathematics" of *ca.* 1895; see [75, p. 7]) "Logic is the science which examines signs, ascertains what is essential to being sign and describes their fundamentally different varieties, inquires into the general conditions of their truth, and states these with formal accuracy, and investigates the law of development of thought, accurately states it and enumerates its fundamentally different modes of working", while what he called "critic" is that part of logic which is concerned explicitly with deduction, and is, thus, a calculus. This suggests, to me at least, that for Peirce logic is both a calculus (as critic) and a language (as semiotic theory); a calculus in the narrow sense, a language in the broader sense.

7 Peirce's distinguishing singular propositions, such as "Socrates is mortal", from universal propositions such as "All Greeks are mortal"

The problem of distinguishing singular from universal propositions was one of the primary, if not the primary, initial motivation for Peirce in undertaking his work in "On an Improvement in Boole's Calculus of Logic" [63]. That work had the goal of improving Boole's algebraic logic by developing a quantification theory which would introduce a more perspicuous and efficacious use of universal and existential quantifiers into Boole's algebra and likewise permit a

[19] My emphasis. Schröder [89, p. 83] first writes: "Am wirksamsten mochte aber zur Richtigstellung der Ansichten die begrundete Bemerkung beitragen dass die Frege'sche 'Begriffsschrift' gar nicht so wesentlich von Boole's Formelsprache sich wie die Jenaer Recension vielleicht auch der Verfasser ausgemacht annimmt", and then [89, p. 84]: "Diesem ihrem Hauptinhalte nach konnte man die Begriffsschrift geradezu eine Umschreibung der Booleschen Formelsprache nennen..."

[20] I owe this historical point to Nathan Houser.

clear distinction between singular propositions and universal propositions.

That work comes to full fruition in 1885 with the Mitchell–Peirce notation for quantified formulas with both indexed (as we discussed in consideration of Peirce's quantification theory based on a system of axioms) and inference rules (see also [54]).

Nevertheless singular propositions, especially those in which definite descriptions rather than proper names occur, have also been termed "Russellian propositions" so-called because of their designation by Russell in terms of the iota quantifier or iota operator employing an inverted iota to be read as "the individual x"; thus, e.g., $(\imath x)(x)$ (see [111, I, p. 32]), and we have, e.g.: "Scott $=$ $(\imath x)(x$ wrote Waverley)".

In *Principia Mathematica* [111, I, p. 54] Whitehead and Russell write "$\phi!x$" for the first-order function of an individual, that is, for any value of the variable which involves only individuals; thus, for example, we might write "μ!"(Socrates) for "Socrates is a man". In the section on "Descriptions" of *Principia* [111, I, p. 181] the iota operator replaces the notation "$\phi!x$" for singulars with "$(\imath x)\Phi(x)$" so that one can deal with definite descriptions as well as names of individuals.

8 On the relations between the algebraic logicians and the "logisticians"

The concept of a distinction between logic as calculus and logic as language was briefly remarked by Russell's student Philip Edward Bertrand Jourdain (1879—1919) in the "Preface" [45] to the English translation [17] by Lydia Gillingham Robinson (1875–?) of *L'algèbra de la logique* [16] of Louis Couturat (1868—1914), a work which fell into the former group, and of the dual development of symbolic logic along these two lines; but Jourdain also admits that the line of demarcation between logicians, such as Boole, De Morgan, Jevons, Venn, Peirce, Schröder, and Ladd-Franklin, working in the aspect of symbolic logic as a *calculus ratiocinator*, and those, the "logisiticans", such as Frege, Peano, and Russell, working in its aspect as a *lingua characteristica* is neither fixed nor precise. He wrote [32, p. iv]: "We can shortly, but very fairly accurately, characterize the dual development of the theory of symbolic logic during the last sixty years as follows: The calculus ratiocinator aspect of symbolic logic was developed by Boole, De Morgan, Jevons, Venn, C. S. Peirce, Schröder, Mrs. Ladd-Franklin and others; the lingua characteristica aspect was developed by Frege, Peano and Russell. Of course there is no hard and fast boundary-line between the domains of these two parties. Thus Peirce and Schröder early began to work at the foundations of arithmetic with the help of the calculus of relations; and thus they did not consider the logical calculus merely as an interesting branch of algebra. Then Peano paid particular attention to the calculative aspect of his symbolism. Frege has remarked that his own symbolism is meant to be a *calculus ratiocinator* as well as a *lingua characteristica*, but the using of Frege's symbolism as a calculus would be rather like using a three-legged stand-camera for what is called "snap-shot" photography, and one of the outwardly most noticeable things about Russell's work is his combination of the symbolisms of Frege and Peano in such a way as to preserve nearly all of the merits of each". Jourdain's reference to "'snap-shot' photography" might well put us in mind of Peirce's comparison of his work in logic with that of Russell when he wrote (see [72, p. 91]) that: "My analyses of reasoning surpasses in thoroughness all that has ever been done in print, whether in words or in symbols — all that De Morgan, Dedekind, Schröder, Peano, Russell, and others have done — to such a degree as to remind one of the differences between a pencil sketch of a scene and a photograph of it".

There is little doubt that Peirce was aware of Frege's work. We know that, at the very least, Christine Ladd-Franklin included (at [46, pp. 70-71]) Frege's *Begriffsschrift* in the bibliography of her "On the Algebra of Logic" for *Studies in Logic* [46] edited by Peirce; that his student Allan Marquand owned a copy of Frege's *Begriffsschrift*; and that the Johns Hopkins University library owned a copy, acquired on 5 April 1881, while Peirce was on the Hopkins faculty; that Peirce received an offprint of Schröder's review of the *Begriffsschrift* [89], and that it has a note in green pencil on it in Peirce's hand: "Formal Logic"[21]; and it is held that Pierce may have sent someone at the University of Jena — where Frege was on the faculty —

[21]See [37], [38], [39, p. 134–137] for discussion of what is and is not known about the interactions between Peirce and Frege.

offprints of his own work, but it is unclear whether in fact he did so, or to whom[22].

How, then, shall we characterize the relation between the "Booleans" and the "Fregeans"? More concretely, how characterize their respective influences upon one another and specifically between Peirce and Frege or relative independence of their achievements in logic?

Randall R. Dipert [22] noted that Peirce was not averse to employing numerical values, not just 1 and 0, for evaluating the truth of propositions, rather than *true* and *false*, but a range of numerical values. He also noted that the formulas employed by Peirce were depending upon the context in which they occurred allowed to have different interpretations so that their terms might represent classes rather than propositions; and hence it would be over-simplifying the history of logic to argue that Peirce was a precursor in these respects of Frege, or anticipated Frege or someone else, certainly not directly, the more so since, whatever Frege knew about Peirce, he first learned belatedly and second-hand through Schröder's numerous references to Peirce in the *Algebra der Logik*.

The heart of the matter for us is to attempt to assess the question of how Peircean the "Fregean revolution" in logic. That is: to what extent did Peirce (and his students and adherents) obtain those elements that characterize the "Fregean" revolution in logic? Our reply must be: "To a considerable extent" but not necessarily all at once and in one particular publication.

To this end, we would do well to borrow the assessment of Jay Zeman who wrote [116, p. 1] that: "Peirce developed independently of the Frege–Peano–Russell (FPR) tradition all of the key formal results of that tradition. He did this in an algebraic format similar to that employed later in *Principia Mathematica*..."

Our account of the criteria and conditions that van Heijenoort set forth as the defining characteristics of modern mathematical logic that have been credited to Frege and in virtue of which Frege is acclaimed the originator, and hence for which he has been judged to be the founder of modern mathematical logic provides substantiation for the assertion by Zeman that Peirce and his coworkers achieved

[22]I owe this point to N. Houser but neither of us have as yet been able to discover the details.

substantially all if not all in the same precise articulation and formulation as Frege, nor everything within the confines of a single work or a single moment. What can be asserted is that over the period of the most productive span of his lifetime as a researcher into formal logic, effectively between the mid-1860s to mid-1890s, Peirce, piecemeal and haltingly, achieved very similar if not quite the same results as did Frege, the latter primarily, but not exclusively, within the confines of his *Begriffsschrift* of 1879. But throughout this period and well into the next it was the work in logic of Peirce and his coworkers, especially Schröder, that dominated the field and that influenced and continued to influence workers in mathematical logic up until Russell, first slowly, with his *Principles of Mathematics*, and Whitehead and Russell together, then expansively, in particular with the appearance in the mid-1920s of the second edition of their *Principia Mathematica*, took the field from the "Booleans" and consummated the "Fregean revolution" in logic. A reassessment of the accomplishments of Peirce's contributions to, and originality in, logic has taken place in recent years in which Hilary Putnam was a leading figure (see [78]), and in which Quine came to participate (see [82] and [83]), and it has been shown (see, e.g. [3]) that much of the work that Russell arrogated to himself (and some of which he attributed to Frege or Peano) not only can be found in Peirce's publications.

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