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CLASSICAL MULTIPLICATIVE LINEAR LOGIC □ INTUITIONISTIC MLL

Abstract. *It is known how to present every deduction in the* {!, I}*-free Classical Multiplicative Linear Logic as (the result of an obvious translation of) a deduction in the intuitionistic* MLL. *We extend the result to the language with I and give short proofs which do not use proof nets.*

1. Introduction

One of the most important computational interpretations of logical proofs uses intuitionistic logic and Curry-Howard isomorphism between natural deduction and lambda terms. One of the goals of linear logic [2] was to provide an improved proof-theoretic model of computation which ensures uniqueness of the normal form of a derivation by means of a new formalism of *proof nets*, which works even for classical linear logic and provides a lot of symmetry. More traditional computational interpretation uses intuitionistic linear logic (cf. [3]) which admits a form of Curry-Howard isomorphism [7]. The results in the literature [10, 1] show how to present every deduction in the {!, I}-free Classical Multiplicative Linear Logic as (the result of an obvious translation of) a deduction in the intuitionistic MLL. We extend the result to the language with *I* and give short proofs which do not use proof nets.

Let us remind that the most important applications of linear logic in algebra depend on the language of MLL with the constant I, cf. [5, 6, 8].

Formulas of the !-free Classical Multiplicative Linear Logic CMLL are constructed from literals (propositional variables p, q, p', …, constant I and their negations $\bar{p}$, $\bar{I}$) by the tensor product ⊗ and *par* connective ℘. Derivable objects of CMLL are *sequents,* i.e. multisets of formulas. CMLL is axiomatized as follows.

Axioms $\bar{p}, p$ $\bar{I}, \ldots, \bar{I}, \bar{p}, p$, I

Inference rules

$$\otimes \quad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \qquad\qquad ℘ \quad \frac{\Gamma, A, B}{\Gamma, A ℘ B}$$

Formulas of the Intuitionistic Multiplicative Linear Logic IMLL are constructed from propositional variables and constant I by linear implication ô and tensor product ⊗. Derivable objects of IMLL are *sequents* $\Gamma \Rightarrow A$, where Γ is a multiset of formulas and A is a formula. IMLL is axiomatized as follows.

Axioms $p \Rightarrow p$ $I, \ldots, I, p \Rightarrow p$, $\Rightarrow I$

Inference rules

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \otimes B} \qquad \frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow C}{A \text{ô} B, \Gamma, \Delta \Rightarrow C}$$

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \text{ô} B} \qquad \frac{A, B, \Gamma \Rightarrow C}{A \otimes B, \Gamma \Rightarrow C}$$

We prove (Theorem 2 below) that every deduction in the classical MLL is (the result of an obvious translation of) a deduction in the intuitionistic MLL up to natural isomorphisms

$$p \otimes I \,\square\, p \qquad\qquad I \,\square\, \bar{I} \wp I \qquad\qquad (1)$$

and *involution*, i.e. interchanging p and $\bar{p}$ for some variables p. This suggests using ordinary lambda-terms to describe CMLL since Curry-Howard isomorphism holds for usual typed lambda-terms and IMLL [7]. Our translation from CMLL into IMLL has an inverse * described in the section 3. Both of them completely preserve the structure of the derivation tree. This shows that every derivation in CMLL is essentially a derivation in IMLL. Moreover, one can fix the goal formula in an arbitrary way. This constitutes one of the differences with the negative translation of the traditional classical propositional logic into intuitionistic logic. The negative translation adds new negations with the corresponding antecedent and succedent rules and levels down important distinctions in the original formula.

This paper incorporates some suggestions of the referee of a previous version.

2. Reduction of CMLL to the balanced I-free fragment

A formula or sequent is *balanced* if each propositional variable occurs there exactly twice, once positively, once negatively. An *instance* of a formula or derivation is a result of substituting some propositional variables by formulas. I-*instance* is obtained when all these formulas are just I. The following well-known proposition (cf. [5]) provides a reduction to balanced sequents.

Lemma 1 *Every derivation d in CMLL is an instance of a derivation of a balanced sequent.*

Proof. Every occurrence of a propositional variable in the last sequent of d is traceable to a unique occurrence in a unique axiom of d. Replace occurrencies traceable to different occurrencies of axioms by distinct variables. □

Note. $p \rightarrow p \& p$ is a counterexample for additive linear logic and the classical propositional calculus.

The next reduction eliminates multiplicative constants.

Lemma 2 *Every derivation in CMLL is an I-instance of an I-free derivation of a balanced sequent up to transformations.*

Proof. Consider given derivation d: Γ of a balanced sequent Γ in CMLL. Every occurrence of I in Γ is traceable to a unique occurrence of I in an axiom. If it comes from one of the first occurrences of $\overline{I}$ in an axiom $\overline{I}, \ldots, \overline{I}, \overline{p}, p$, replace these occurrences of $\overline{I}$ by $\overline{q_1}, \ldots, \overline{q_n}$ and the last p by $p \otimes q_1 \ldots \otimes q_n$ for distinct fresh variables $q_1, \ldots, q_n$ (and make the same replacement for all occurrences traceable to these). If it is one of the last two occurrences in $\overline{I}, \ldots, \overline{I}, \overline{I}, I$ or in $\overline{I}, I$, replace both occurrences by a fresh variable q. If it is an occurrence in an axiom I, replace it by $\overline{q} \wp p$ for a fresh q. □

3. Derivations of IMLL-sequents

Consider the standard translation of IMLL into CMLL:

$$(\Gamma \Rightarrow \Delta)^* := \overline{\Gamma}, \overline{\overline{\Delta}}$$

where

$$\overline{A \otimes B} := A \wp B; \quad \overline{A \multimap B} := A \otimes B; \quad \overline{\overline{A}} := A; \quad \overline{A \wp B} := \overline{A} \otimes \overline{B}$$

and induced translation of derivations. Double negation over Δ is inserted to replace the linear implication $A \multimap B$ by $\overline{A} \wp B$. As a warm-up consider the case when involution is not needed. Note that Δ below is allowed to contain arbitrary many formulas.

Theorem 1 *If $\Gamma\Delta$ are multisets of formulas of IMLL and d: $(\Gamma \Rightarrow \Delta)^*$ is a derivation in CMLL, then Δ consists of one formula and $d \equiv e^*$ for some e: $\Gamma \Rightarrow \Delta$ in IMLL.*

Proof. Induction on d. If d is an axiom $\overline{p}, p$ or $\overline{I}, \overline{p}, p$ then $\Delta \equiv p$ and e is $p \Rightarrow p$ or $I, p \Rightarrow p$. If *lastrule(d)*=$\otimes$ then

$$\frac{\overline{\Gamma}_1, \Delta_1', A \quad \overline{\Gamma}_2, \Delta_2', B}{d: \ \overline{\Gamma}, \Delta', A \otimes B} \quad \text{or} \quad \frac{\overline{\Gamma}_1, \Delta_1, A \quad \overline{B}, \overline{\Gamma}_2, \Delta_2}{d: \ A \otimes \overline{B}, \overline{\Gamma}, \Delta}$$

In the first case $\Delta_1' \equiv \Delta_2' \equiv \varnothing$ by the induction hypothesis (IH), and one has e: $\Gamma \Rightarrow A \otimes B$. In the second case $\Delta_1 \equiv \varnothing$ and $\Delta_1 \equiv \Delta$ consists of one formula, so that e: $A \multimap B, \Gamma \Rightarrow \Delta$:

$$\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{e:\ \Gamma \Rightarrow A \otimes B} \quad \text{or} \quad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, B \Rightarrow \Delta}{e:\ A\text{ô}B, \Gamma \Rightarrow \Delta}$$

Finally, if the *lastrule(d)*= $\wp$, one has:

$$\frac{\overline{\Gamma}, \Delta, \overline{A}, B}{d:\ \overline{\Gamma}, \Delta, \overline{A} \wp B} \qquad \frac{\Gamma, A \Rightarrow B}{e:\ \Gamma \Rightarrow A\text{ô}B}$$

since $\Delta_1 \equiv \varnothing$, or

$$\frac{\overline{A}, \overline{B}, \overline{\Gamma}, \Delta}{d:\ \overline{A} \wp \overline{B}, \overline{\Gamma}, \Delta} \qquad \frac{A, B, \Gamma \Rightarrow \Delta}{e:\ A \otimes B, \Gamma \Rightarrow \Delta}$$

□

4. General case

Theorem 2 *For every balanced I-free sequent of CMLL and its derivation d: Σ, C in CMLL there are formulas (intuitionistic translations) Σ_1, C_1 in IMLL, an involution ι and a deduction e: $\Sigma_1 \Rightarrow C_1$ in IMLL (all depending of the choice of C) such that $d \equiv e^{*\iota}$.*

Proof. Induction on d. The case d≡p, $\overline{p}$ is obvious. Consider subcases depending of the *lastrule(d)* and a position of C. Let *lastrule(d)*=⊗ and C is the principal formula:

$$\frac{f: \Gamma, A \quad g: \Delta, B}{d: \Gamma, \Delta, A \otimes B}$$

Then by IH there are $f_1: \Gamma_1 \Rightarrow A_1$, $g_1: \Delta_1 \Rightarrow B_1$ and involutions ι′, ι″ such that $f \equiv f_1^{*\iota'}$, $g \equiv g_1^{*\iota''}$. Note that propositional variables in the premises are distinct, and define: $\iota := \iota' \cup \iota''$,

$$\frac{f_1: \Gamma_1 \Rightarrow A_1 \quad g_1: \Delta_1 \Rightarrow B_1}{e:\ \Gamma_1, \Delta_1 \Rightarrow A_1 \otimes B_1}$$

Remaining cases are similar.

$$\frac{\Gamma, A, C \quad \Delta, B}{d: \Gamma, \Delta, A \otimes B, C} \qquad \frac{\Gamma_1, \overline{A}_1 \Rightarrow C_1 \quad \Delta_1 \Rightarrow B_1}{e: \Gamma_1, \Delta_1, B_1\text{ô}\ \overline{A}_1 \Rightarrow C_1}$$

$$\frac{\Gamma, A, B}{d: \Gamma, A \wp B} \qquad \frac{\Gamma_1, \overline{A}_1 \Rightarrow B_1}{e: \Gamma_1 \Rightarrow \overline{A}_1\text{ô}B_1} \quad \text{or} \quad \frac{\Gamma_1, \overline{B}_1 \Rightarrow A_1}{e: \Gamma_1 \Rightarrow \overline{B}_1\text{ô}A_1}$$

In this case one can choose arbitrarily between two possible "witchings" of $A \wp B$.

$$\frac{\Gamma,\ A,\ B,\ C}{d: \Gamma,\ A \wp B,\ C} \qquad \frac{\Gamma_1,\ \bar{A}_1,\ \bar{B}_1 \Rightarrow C_1}{e: \Gamma,\ \bar{A}_1 \otimes \bar{B}_1 \Rightarrow C_1}$$

□

Conclusion

Let us sum up.

For every derivation d: C in CMLL one has a derivation *Int(d)*:$\Rightarrow C_1$ in IMLL and an involution ι such that d□$(Int(d))^{*\iota}\theta$ and C□$C_1^{*\iota}\theta$ for some substitution θ.

Moreover, for I-free balanced formula C the derivation *Int(d)* depends only of C.

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